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UNIVERSITY OF ALBERTA

POSITIVE DECAYING SOLUTIONS OF NONLINEAR ELLIPTIC PROBLEMS

by

LAO SEN YU ©

A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled POSITIVE DECAYING SOLUTIONS OF NONLINEAR ELLIPTIC PROBLEMS submitted by LAO SEN YU in partial fulfillment of the degree of Doctor of Philosophy.

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TO MY PARENTS AND MY WIFE

ABSTRACT

In this thesis, we consider the existence of positive decaying solutions to subcritical nonlinear elliptic problems on unbounded domains. Specifically, we examine the equation $\ell u = f(x,u)$ and the associated eigenvalue problem $\ell u = \lambda f(x,u)$, where ℓ represents one of the elliptic operators formally given by:

$$\ell u = \begin{cases} -\sum_{i,j=0}^{n} D_i(a_{ij}(x)D_j u) + b(x)u, \\ \\ -\operatorname{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big) + b(x)|u|^{p-2}u, \\ \\ (-\Delta + b'_m)(-\Delta + b'_{m-1})\cdots(-\Delta + b'_1)u, \end{cases}$$

and $f(x,t)$ has subcritical growth at $t = \infty$.

While we obtain several extensions of earlier results, our main motivation for this thesis comes from open questions which have been recently posed in the literature by Kusano, Naito and Swanson [KNS3] and Noussair and Swanson [NS4]. Specifically, these are:

(a) The existence of positive decaying solutions of the superlinear polyharmonic problem;

(b) The relaxation of the condition $\overline{\lim_{|x|\to\infty}} \frac{f(x,t)}{t} = 0$ uniformly on $[0,T]$ for the existence of a pair of positive decaying solutions.

We consider these questions in this thesis. Basically our procedure involves a variational approach on functionals defined on weighted Sobolev spaces suitable for our problems. We apply Mountain Pass Theorem arguments to establish the existence of solutions. We then make use of various L^p-estimates to show the decay. In this way, we are able to treat problems with highly nonradial coefficients. We consider the sharpness of our conditions both in the nonradial and radial cases by comparison with earlier results.

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CHAPTER 1

INTRODUCTION

For their practical applications and mathematical importance, elliptic equations have attracted intensive attention from mathematicians for several decades. In recent years considerable interest has been focused on the second order semilinear elliptic problem

$$\ell u \equiv -\sum D_i(a_{ij}(x)D_j u) + b(x)u = f(x,u) \tag{1.1}$$

and the associated eigenvalue problem

$$\ell u \equiv -\sum D_i(a_{ij}(x)D_j u) + b(x)u = \lambda f(x,u). \tag{1.2}$$

These two problems arise in many fields, such as physics, chemistry and biology. For example, the so-called Emden-Fowler equation: $-\Delta u = K(x)u^\alpha$ $(\alpha \geq 0)$ - a special case of (1.1) - occurs in the study of gas dynamics, fluid mechanics and chemically reacting systems, see Wong [Wo] and references therein. In each of these models, u has a specific physical meaning. In particular, Matukuma's equation $-\Delta u = \frac{1}{1+|x|^2}u^\alpha$ $(\alpha > 1)$ is used to model globular clusters of stars. Here $u > 0$ represents the gravitation potential, see [Ni2]. (1.1) also arises in the search for certain kinds of solitary waves in the nonlinear Klein-Gordon equation:

$$\Phi_{tt} - \Delta\Phi + a^2\Phi = f(\Phi)$$

and nonlinear Schrödinger equation:

$$i\Phi_t - \Delta\Phi = f(\Phi).$$

Specifically, looking for solitary waves of "standing wave" type, that is $\Phi(t,x) = e^{i\omega t}u(x)$ for the Klein-Gordon equation and $\Phi(t,x) = e^{-i\omega t}u(x)$ for the Schrödinger equation, leads to:

$$-\Delta u + (a^2 - \omega^2)u = f(u)$$

and:

$$-\Delta u + \omega^2 u = f(u),$$

if $f(e^{i\theta}u) = e^{i\theta}f(u)$. Equation (1.2) also arises in the study of the temperature distribution in an object heated by the application of electric current:

$$u_t - \nabla(A(x)\nabla u) = \lambda f(x,u),$$

where u is the temperature distribution, $\lambda^{\frac{1}{2}}$ is the current, and f is the electric resistance. The steady states lead to (1.2).

On the other hand, special cases of equation (1.1) are motivated by Riemann geometry. More precisely, the equation

$$-\frac{4(n-2)}{n-2}\Delta_g u + ku = K(x)u^{\frac{n+2}{n-2}},$$

where $\Delta_g = \frac{1}{\sqrt{|g|}}\sum_{i,j=1}^{n} D_i(\sqrt{|g|}\ g^{ij}D_j)$ with $|g| = \det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$, is known as the conformal scalar curvature equation. For more details, we refer to [Ni2].

There have been many studies of equations (1.1) and (1.2) on both bounded and unbounded domains. For the bounded domain case we mention from the available literature only results important to this thesis. We recall, specifically, the Mountain Pass Theorem by Ambrosetti and Rabinowitz [AB] and its applications, the result for the critical case by Brezis and Nirenberg [BN] and the nonexistence theorem by Pohozaev [Po]. Due to these and other contributions, much is known about the structure of the solutions of (1.1) (or (1.2)) on bounded domains. For unbounded domains, the case of special interest here, there are also a great number of papers. We mention, in particular, the results of Atkinson and Peletier [AP], Berestycki and Lions [BL], Ding and Ni [DN1, DN2]. Gidas, Ni and Nirenberg [GNN], Gidas and Spruck [GS], Kawano, Satsuma and Yotsutani [KSY], Li and Ni [LN1, LN2], Lions [Li1, Li2], Ni [Ni1, Ni2], Ni and Serrin [NS1, NS2], Strauss [St] and the references therein. Most of these results were obtained for the radially symmetric case: $\Omega = R^n$, $\ell = -\Delta + b$ (b a nonnegative number), $f(x,t) = f(|x|,t)$ or $f(x,t)$ admits suitable radial majorants. The methods involved the application of ODE theory, and the combination of variational calculus and radially symmetric rearrangement. Only relatively few papers considered the nonradial case. See [Al] by Allegretto, [AH1] by Allegretto and Huang, [CV] by Chaljub-Simon and Volkmann and [NS1, NS2, NS3] by Noussair and Swanson. In [Al] and [AH1], the existence of solutions bounded above and below by positive constants was proved through fixed point theorem arguments. In [CV] and [NS1], a positive decaying solution was obtained for the equation with prototype:

$$-\sum D_i(a_{ij}(x)D_ju) + b(x)u = g(x)u^\alpha, \quad x \in \Omega$$

Ω unbounded, $1 < \alpha < \frac{n+2}{n-2}$, by means of weighted Sobolev spaces in [CV] and by subdomain approximation arguments in [NS1]. Although the conditions on f in the two papers are different, both assume $b(x) \geq b_0 > 0$, which is a significant assumption in the case of Ω unbounded. In order to relax $b(x) \geq b_0 > 0$, Noussair and Swanson considered the more general case $b(x) \geq 0$ in [NS2, NS3] and obtained a positive decaying solution for

$$-\Sigma D_i(a_{ij}(x)D_ju) + b(x)u = \lambda f(x,u)$$

for $\lambda = \lambda_0$, some $\lambda_0 > 0$, and f with suitable radial majorant. Rescaling techniques can eliminate the eigenvalue parameter λ in front of $f(x,u)$, provided that $f(x,t)$ is homogeneous in t. In other words, the relaxation of $b(x) \geq b_0 > 0$ introduced a Lagrange multiplier $\lambda = \lambda_0$ for nonhomogeneous f. Therefore unlike the bounded domain case, the structure of the solutions of semilinear elliptic equations on unbounded domains is not so clear, even though many efforts have been made. This difference is mainly caused by three aspects: the failure of the L^2-theory, of the Poincare Inequality and of the Compact Embedding Theorem. A consequence of this failure is that generally the usual Sobolev space $W_0^{1,2}(\Omega)$ is no longer a suitable function space in which to seek our solutions. Positive decaying solutions of $\ell u = f(x,u)$ may not be in $W_0^{1,2}(\Omega)$

due to the fact that $|x|^{2-n}$ is a subsolution of $-\Delta u = g(x)u^{\alpha}$ $(\alpha \geq 0)$; the norm $(\ell u,\ u)^{\frac{1}{2}}$ induced by operator ℓ when $b(x) \not\geq b_0 > 0$ may not be a equivalent norm in $W_0^{1,2}(\Omega)$; even with $b(x) \geq b_0 > 0$, $\int_\Omega f(x,u)\varphi\ dx$ generally does not define a compact map from $W_0^{1,2}(\Omega)$ to $W_0^{1,2}(\Omega)$.

To overcome these difficulties, we must find some special function spaces suitable for our problems, and somehow set up the compactness of the map $\int_\Omega f(x,u)\varphi\ dx$. In this thesis, we study the subcritical semilinear elliptic problems given by:

$$\begin{cases} \ell u = f(x,u), & x \in \Omega \\ u|_{\partial\Omega} = 0, & \lim\limits_{|x|\to\infty} u = 0 \end{cases} \tag{1.4}$$

where Ω is a possibly unbounded domain, $f(x,t)$ could be pure sublinear, pure superlinear or mixed sublinear and superlinear. We also consider the problem:

$$\begin{cases} \ell u = \lambda f(x,u), & x \in \Omega \\ u|_{\partial\Omega} = 0, & \lim\limits_{|x|\to\infty} u = 0 \end{cases} \tag{1.5}$$

where $f(x,t)$ is superlinear and odd in t or has multiple "humps" in t. We obtain criteria for the existence of positive decaying solutions for (1.4) and (1.5) without any assumptions of radial symmetry or $b(x) \geq b_0 > 0$. Our tools are weighted spaces and various variants of the Mountain Pass Theorem. While our results cover the case with highly nonradial properties, we show that even in the radial case, they are equivalent to the earlier optimal results in some sense. We do not require $f(x,t)$ to be homogeneous in t for problem (1.4).

Our method for handling (1.4) and (1.5) is then extended to consider the study of the higher order semilinear elliptic problem:

$$\begin{cases} \sum_{i=0}^{m} b_i(-\Delta)^i u = f(x,u), \quad x \in \Omega \\ D^{\nu} u|_{\partial\Omega} = 0, \quad \lim_{|x|\to\infty} D^{\nu} u = 0 \end{cases} \tag{1.6}$$

where ν is multi-integer with $0 \le |\nu| \le m-1$, and b_i is a nonnegative number with $b_m = 1$. The equation

$$\sum_{i=0}^{m} b_i(-\Delta)^i u = f(x,u) \tag{1.7}$$

does not seem to have as much physical meaning as the second order elliptic equation, and thus is essentially of theoretical interest. There are relatively few studies of (1.7) on unbounded domains. The first papers are due to Walter [Wa1, Wa2] and Walter and Rhee [WR], where $\Delta^m u = e^u$ and $\Delta^m u = f(|x|, u)$ in R^n were studied. Since 1989, (1.7) has received more attention. See the references by Allegretto and Huang [AH2], Bernis [Ber], Dalmasso [Da1, Da2], Edelson [Ed], Fukagai [Fu2], Kusano, Naito and Swanson [KNS1, KNS2, KNS3, KNS4], Kusano and Swanson [KS] and Usami [Us]. Most of the papers employed radially symmetric arguments, that is: $\Omega = R^n$, $f(x,u) = f(|x|, u)$ (or $f(|x|, \Delta u, \ldots, \Delta^{m-1} u)$). The existence and behaviour at ∞ of solutions were thus investigated via an ODE approach. In this way, the existence of positive decaying solutions in radial cases was partially solved. In [KNS3] (see also [KS]), Kusano, Naito and Swanson obtained a positive decaying solution for $f(x,t)$ sublinear or mixed sub-superlinear, and gave the pure superlinear case as an open question. Dalmasso [Da1] partially answered this open question in the special case $m = 2$, that is, he solved $\Delta^2 u = g(|x|)u^\alpha$ $(1 < \alpha < \frac{n+4}{n-4})$ in R^n, $n \geq 5$. Therefore the existence of positive decaying solutions for general $f(x,t)$ basically remains open. In this thesis we give a complete answer to this question. Specifically we consider (1.6) for f sublinear, superlinear and mixed sub-superlinear under general nonradial conditions on f, which, as shown by some examples, are better than the former ones for radial cases. While the existence of a solution to the equation can be fairly easily proved by expanding the second order approach, the key is the proof of the decay of the solution. We employ L^p-estimates by Agmon [Ag] for this purpose.

As further application, we slightly modify the same method to study the p-Laplacian problem:

$$\begin{cases} -\Delta(a(x)|\nabla u|^{p-2}\nabla u) + b(x)|u|^{p-2}u = f(x,u) \quad \text{in} \quad \Omega \\ u|_{\partial\Omega} = 0, \quad \lim\limits_{|x|\to\infty} u = 0 \end{cases} \tag{1.8}$$

with $1 < p < n$. When $p = 2$, (1.8) is a second order elliptic problem. Like the second order elliptic problem, (1.8) has many physical applications. It is used to model reaction-diffusion, flow through porous media, petroleum extraction and so on, see Diaz [Di].

For the general case $1 < p < n$, many studies have appeared recently. We refer to the work in the bounded domain case by Azorero and Alonso [AA], Egnell [Eg1, Eg2], Guedda and Veron [GV], Kichenassamy and Veron [KV], Veron [Ve] and references therein. In the unbounded domain case, there have been several papers, see Li and Yan [LY] and Ni and Serrin [NiS1, NiS2]. But no existence theory seems to have been found for the nonradially symmetric problem (1.8). We will obtain sufficient conditions for (1.8) to have positive solutions by the same basic approach.

The thesis is organized as follows. In Chapter 2, we set up the notational framework and some weighted spaces suitable for our problems, and analyze the properties of the spaces. In Chapter 3, we study the second order problems (1.4) and (1.5), and establish conditions for (1.4) and (1.5) to have a positive solution or infinitely many solutions via the Mountain Pass Theorem. We adapt a device due to Brezis and Kato [BK] to establish the estimate: $u \in L^p(\Omega)$ for $\frac{2n}{n-2} \leq p < \infty$. The decay of u follows immediately from this estimate. Chapter 4 is devoted to the study of the higher order problem (1.6), and the method in Chapter 3 is slightly modified to prove the existence of a solution. Since Brezis and Kato's iteration technique does not seem to work for higher order problems, we make use of the L^p-estimate of Agmon [Ag] to show the decay of the solution. In the special case $\Omega = R^n$, we show that the solution obtained must be positive. But for general Ω, we can not guarantee the positivity of the solution obtained due to the lack of enough information on u at the boundary $\partial\Omega$. This is one of the differences between second order and higher order problems. In Chapter 5, we prove some multiple solution existence results for second order eigenvalue problems when f has multiple "humps". As an application, Chapter 6 modifies the method of Chapter 3 to study the p-Laplacian problem and obtains existence criteria. Chapter 7 concludes the thesis with remarks and open questions.

CHAPTER 2

BACKGROUND MATERIALS

2.1. Introduction.

The purpose of this chapter is to set up the notational framework, which will be used in the thesis, and some weighted Sobolev spaces suitable for our problems. There are many reasons for us to introduce spaces other than usual Sobolev spaces. When $b = 0$, the L^2-theory of the following equation

$$\ell u \equiv -\Delta u + bu = g(x)u^{\alpha}, \;\; x \in R^n, \;\; \alpha > 1 \qquad (2.1.1)$$

breaks down. In fact, (2.1.1) when $b = 0$ does not have any positive solutions u in $L^2(R^n)$ for $n = 3, 4$, because of the well-known fact that any positive solution $u(x)$ of $-\Delta u \geq 0$ for $|x| \geq 1$ is bounded below by $C|x|^{2-n}$ for $|x| \geq 1, \; n \geq 3$. Note that $|x|^{2-n} \notin L^2(|x| \geq 1)$ for $n = 3, 4$. On the other hand, due to lack of Poincare-type inequalities in unbounded domains, the "norm"

$$\left(\int_{R^n} |\nabla u|^2 + bu^2 \right)^{\frac{1}{2}} \qquad (2.1.2)$$

induced by ℓ is in general no longer an equivalent norm in usual Sobolev space $W_0^{1,2}(R^n)$, so that we can not consider (2.1.1) by simply using a variational method. Therefore we seek some spaces in which (2.1.2) is an equivalent norm and the local properties of the usual Sobolev spaces still hold.

This chapter is organized as follows: in Section 2.2 we set up notations, in Section 2.3 we give two Hardy-type inequalities, which are used in the construction of weighted Sobolev spaces. We then introduce in Section 2.4 these spaces and analyze their properties. Finally, in Section 2.5, we give the versions of the Mountain Pass Theorem which are employed to obtain the existence of the solutions of our problems.

2.2. Notations.

We denote by R^n the n-dimensional Euclidean space, by Ω a smooth subdomain (open, connected set) of R^n. For any Banach space, we denote by $B_r(x)$ the ball of radius r centered at x, with $B_r \triangleq B_r(0)$. We set $\Omega_r = \Omega \cap B_r$, where $B_r \subseteq R^n$.

We denote by ℓ one of the following three operators:

(i) the second order elliptic operator defined by

$$\ell u = -\sum_{i,j=1}^{n} D_i(a_{ij}(x) D_j u) + b(x)u \tag{2.2.1}$$

where $D_i = \frac{\partial}{\partial x_i}$, $a_{ij}(x) = a_{ji}(x) \in C^{1+\alpha}_{\text{loc}}(\overline{\Omega}) \cap L^\infty(\overline{\Omega})$, $0 \le b(x) \in C^{\alpha}_{\text{loc}}(\overline{\Omega}) \cap L^\infty(\overline{\Omega})$. We always assume that this second order elliptic operator ℓ is uniformly elliptic, that is,

$$\nu|\xi|^2 \le \sum a_{ij}\xi_i\xi_j \le \delta|\xi|^2$$

for $\xi \in R^n$ and some positive numbers $0 < \nu \le \delta$,

(ii) the higher order elliptic operator defined by

$$\ell u = \sum_{i=0}^{m} b_i(-\Delta)^i u \tag{2.2.2}$$

where Δ is the Laplacian, $m \geq 1$, b_i is a non-negative number, $b_m = 1$. We always assume that ℓ can be factorized as

$$\ell u = \prod_{i=1}^{m}(-\Delta + b_i')u \equiv (-\Delta + b_m')\cdots(-\Delta + b_1')u \tag{2.2.3}$$

where b_i' is also a non-negative number, depending on b_i's. Note that $m = 1$ is a special case of (i). If so, we allow b_0 to be a function: $b_0 = b(x) \geq 0$;

(iii) the p-Laplacian operator defined by

$$\ell u = -\text{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big) + b(x)|u|^{p-2}u \tag{2.2.4}$$

where $1 < p < n$, $0 < a_0 \leq a(x) \in C^1(\overline{\Omega}) \cap L^\infty(\overline{\Omega})$, $0 \leq b(x) \in C^0(\overline{\Omega}) \cap L^\infty(\overline{\Omega})$.

In any specific case, it will be clear from the context whether ℓ denotes (2.1.1), (2.1.2) or (2.1.4).

2.3. Hardy-type Inequalities.

Here we introduce two Hardy-type inequalities, which will furnish the weights in the spaces introduced below. Let $\sigma(x) = (1 + |x|^2)^{-1}$.

LEMMA 2.1. *Let $\varphi \in C_0^\infty(\Omega)$, then for $n > 2i \geq 2$,*

(a) $C \int_\Omega \sigma^i \varphi^2 \leq \int_\Omega \sigma^{i-1} |\nabla\varphi|^2$

(b) $C \int_\Omega \sigma^{i-1} |\nabla\varphi|^2 \leq \int_\Omega \sigma^{i-2} |\Delta\varphi|^2$

where $C = C(n, i)$ is a constant independent of φ and Ω.

PROOF: (a) By the Divergence Theorem:

$$\int_\Omega \sigma^i \varphi = -\frac{1}{n}\int_\Omega x \cdot \nabla(\sigma^i \varphi^2)$$

$$= -\frac{2}{n}\int_\Omega \left[\sigma^i \varphi x \cdot \nabla\varphi - i\sigma^i \frac{|x|^2}{1 + |x|^2}\varphi^2\right]$$

yielding:

$$\begin{aligned}\left(1-\frac{2i}{n}\right)\int_\Omega \sigma^i\varphi^2 &\le \frac{2}{n}\int_\Omega \sigma^{i-\frac{1}{2}}|\varphi|\ |\nabla\varphi|\ \frac{|x|}{\sqrt{1+|x|^2}}\\ &\le \frac{1}{n}\int_\Omega \left(\sigma^{\frac{i}{2}}|\varphi|\right)\cdot\left(\sigma^{\frac{i-1}{2}}|\nabla\varphi|\right).\end{aligned}$$

The first inequality is immediate.

(b) Applying again the Divergence Theorem and observing $\varphi|_{\partial\Omega}=0$, give

$$\begin{aligned}&\int_\Omega \sigma^{i-1}|\nabla\varphi|^2 = -\int_\Omega \varphi\sum_j D_j(\sigma^{i-1}D_j\varphi)\\ &= \int_\Omega \sigma^{i-1}\varphi(-\Delta\varphi) + (i-1)\sum_j\int_\Omega 2\varphi(D_j\varphi)\sigma^i x_j\\ &= \int_\Omega \sigma^{i-1}\varphi(-\Delta\varphi) + (i-1)\int_\Omega\sum_j D_j(\varphi^2)\sigma^i x_j\\ &= \int_\Omega \sigma^{i-1}\varphi(-\Delta\varphi) - (i-1)\int_\Omega\varphi^2\sum_j D_j(\sigma^i x_j)\\ &= \int_\Omega \sigma^{i-1}\varphi(-\Delta\varphi) - (i-1)\int_\Omega\left[n-\frac{2i|x|^2}{1+|x|^2}\right]\sigma^i\varphi^2\\ &\le \int_\Omega \sigma^{i-1}\varphi(-\Delta\varphi) \le \int_\Omega\left(\sigma^{\frac{i}{2}}|\varphi|\right)\left(\sigma^{\frac{i}{2}-1}|\Delta\varphi|\right).\end{aligned}$$

Inequality (b) now follows from the Cauchy-Schwarz Inequality and (a).

2.4. Weighted Sobolev Spaces.

With the aid of Lemma 2.1 and some elementary limit arguments we can set up weighted spaces suitable for our problems. Let $W^m(\Omega)$ denote the space of functions with up to m^{th} weak derivatives. Let $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$

denote the usual Sobolev spaces. In $W^m(\Omega)$ we define a weighted space $\widehat{E}(m)$ as follows

$$\widehat{E}(m) = \{u \in W^m(\Omega) \mid \|u\| < \infty\},$$

where

$$\|u\|^2 = \sum_{i=0}^{m} \int_\Omega \omega_i |\nabla^i u|^2 dx, \tag{2.4.1}$$

$\omega_i = \max\{\sigma^{m-i},\ b_i\}$, b_i as given in (2.2.2), and

$$\nabla^i \equiv \begin{cases} \Delta^j & \text{if } i = 2j, \\ \nabla(\Delta^j) & \text{if } i = 2j+1. \end{cases}$$

$\widehat{E}(m)$ is a Hilbert space. Indeed, let $\{u_j\}$ be a Cauchy sequence in $\widehat{E}(m)$, then $\{\Delta^i u_j\}$ is also a Cauchy sequence in $L^2_{\omega_i(\Omega)}$, $i = 0, \ldots, m$. The completeness of $L^2_{\omega_i(\Omega)}$ implies that $\nabla^i u_j \to v^i$ in $L^2_{\omega_i(\Omega)}$. Denote v_0 by v. By a limit argument $\nabla^i v = v^i$ in the distribution sense. It follows, from Theorem 1 on page 190 of [BJS], that $v \in W^m$. Clearly $\|v\| < \infty$. Thus $v \in \widehat{E}(m)$.

Let $E(m)$ be the closure of $C_0^\infty(\Omega)$ in $\widehat{E}(m)$ with respect to the norm $\|\cdot\|$ defined in (2.4.1). $E(m)$ is also a Hilbert space. For convenience, we simply denote $E(m)$ by E when it is clear in the context. It is useful to consider in detail a special case of E. Let $m = 1$ and note that, by definition, the norm of $E(1)$ now is

$$\|u\|^2 = \int_\Omega |\nabla u|^2 + \omega_0 u^2$$

where $\omega_0 = \max\{\sigma(x),\ b(x)\}$. If, furthermore, Ω is bounded, then $\|\cdot\|$ is equivalent to the usual Sobolev norm $\|\cdot\|_{W^{1,2}}$, and consequently $E(1)$ is equivalent to $W_0^{1,2}(\Omega)$. In fact, for any $m \geq 1$, and bounded Ω, $E(m)$ is equivalent to $W_0^{m,2}(\Omega)$, as we have the following inequalities and identities:

$$c_1\|u\|^2 \leq \sum_i \int_\Omega |\nabla^i u|^2 \leq C_2\|u\|^2,$$

$$\int_\Omega |\nabla^i u|^2 = \sum_{|\alpha|=i} \int_\Omega |D^\alpha u|^2,$$

where $u \in C_0^\infty(\Omega)$, $C_1, C_2 > 0$.

To establish properties of E it is useful to recall

LEMMA 2.2. *Let $v \in W^{1,2}(\Omega_r)$ be a weak solution of $-\Delta v = f$, $v = 0$ on $\partial\Omega$. If $f \in W^{k,2}(\Omega_r)$, then for $r' < r$, we have $v \in W^{k+2,2}(\Omega_{r'})$ and*

$$\|v\|_{W^{k+2,2}(\Omega_{r'})} \leq C\{\|v\|_{L^2(\Omega_r)} + \|f\|_{W^{k,2}(\Omega_r)}\}$$

where $C = C(n,k,r,r')$.

PROOF: We need only modify slightly the proof of [GT, Th. 8.12]. Indeed, the compactness of $\partial\Omega \cap \partial\Omega_r$ gives immediately the $W^{k+2,2}$ estimates on v near $\partial\Omega \cap \partial\Omega_r$. Next we apply the interior estimates [GT, Th. 8.10] to $\Omega_{r'}$. We observe, finally, that $\|v\|_{W^{1,2}}$ in [GT, Th. 8.10] may be replaced by $\|v\|_{L^2}$. See the remark after the proof of [GT, Th.8.8].

We note the following properties of E:

LEMMA 2.3. *For any $u \in E$,*

(a) $\Delta^i u \in W_{\mathrm{loc}}^{1,2}(\Omega)$, for $i = 0, \ldots, N$, where $N : m = 2(N+1)$ or $2N+1$;

(b) $u \in W^{m,2}(\Omega_r)$ *and*

$$\|u\|_{W^{m,2}(\Omega_r)} \le C\|u\|, \quad C = C(n,m,r);$$

(c) $u \in L^{2n/(n-2m)}(\Omega)$ *and*

$$\|u\|_{2n/(n-2m)} \le C\|u\| \quad \textit{for} \quad n > 2m, \ C = C(n,m);$$

(d) $\|\cdot\| \sim \|\cdot\|_\ell$ *where*

$$\|u\|_\ell^2 = \sum_{i=0}^{m} \int_\Omega b_i |\nabla^i u|^2 dx. \tag{2.4.2}$$

Proof: Part (a) is immediate from the definition of E. We need only prove that (b) holds for all $u \in C_0^\infty(\Omega)$ since $C_0^\infty(\Omega)$ is dense in E. Choose $r_0 > r$. If $m = 2(N+1)$, then let $v = \Delta^N u$, $f = \Delta^{N+1} u$. Observing that $v|_{\partial\Omega} = 0$, $f \in L^2(\Omega)$ and applying Lemma 2.2, we obtain that $\Delta^N u \in W^{2,2}(\Omega_{r_1})$ for $r < r_1 < r_0$ and

$$\|\Delta^N u\|_{W^{2,2}(\Omega_{r_1})} \le C\{\|\Delta^N u\|_{L^2(\Omega_{r_0})} + \|\Delta^{N+1} u\|_{L^2(\Omega_{r_0})}\}.$$

Iterating this process yields $u \in W^{2(N+1),2}(\Omega_r)$ and

$$\|u\|_{W^{2(N+1),2}(\Omega_r)} \le C \sum_{i=0}^{N+1} \|\Delta^i u\|_{L^2(\Omega_{r_0})}$$

$$\le C \sum_{i=0}^{N+1} \|\Delta^i u\|_{L^2_{\omega_{2i}}(\Omega)} \le C\|u\|$$

where $C = C(n,m,r)$. If $m = 2N+1$, we start with $v = \Delta^{N-1} u$, $f = \Delta^N u$ with $v|_{\partial\Omega} = 0$ and $f \in W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$. Similarly $\Delta^{N-1} u \in W^{3,2}(\Omega_{r_1})$ for $r < r_1 < r_0$ and

$$\|\Delta^{N-1} u\|_{W^{3,2}(\Omega_{r_1})} \le C\{\|\Delta^{N-1} u\|_{L^2(\Omega_{r_0})} + \|\Delta^N\|_{L^2(\Omega_{r_0})} + \|\nabla\ \Delta^N u\|_{L^2(\Omega_{r_0})}\},$$

the rest of the proof is the same as the case $m = 2(N+1)$, and (b) holds. To prove (c), note that by Sobolev's Inequality, for any $u \in C_0^\infty(\Omega)$

$$\begin{aligned} \|u\|^2_{L^{2n/(n-2m)}(\Omega)} &\leq C \int_\Omega \sum_{|\alpha|=m} |D^\alpha u|^2 \\ &= C \int_\Omega |\nabla^m u|^2 \leq C \|u\|^2 \end{aligned}$$

for some constant C independent of u. Finally to prove (d), observe that $\|u\|_\ell \leq \|u\|$ is obvious. Apply Lemma 2.1 to the leading term of $\|u\|^2$ for a finite number of times:

$$\begin{aligned} \int_\Omega |\nabla^m u|^2 &\geq \frac{1}{2} \int_\Omega |\nabla^m u|^2 + \frac{C}{2} \int_\Omega \sigma |\nabla^{m-1} u|^2 \\ &\cdots\cdots \\ &\geq \frac{1}{2} \sum_{i=0}^{m} \left(\frac{C}{2}\right)^{m-i} \int_\Omega \sigma^{m-i} |\nabla^i u|^2 \\ &\geq C_1 \sum_{i=0}^{m} \int_\Omega \sigma^{m-i} |\nabla^i u|^2. \end{aligned}$$

This yields:

$$\begin{aligned} \|u\|^2_\ell &\geq C_2 \sum_{i=0}^{m} \int_\Omega (\sigma^{m-i} + b_i) |\nabla^i u|^2 \\ &\geq C_3 \|u\|^2. \end{aligned}$$

Part (d) follows.

We note that Lemma 2.3 implies the following important properties which will be used quite often later on:

(a) ℓ induces an equivalent norm and an equivalent inner product:

$$(u,v)_\ell = \sum_{i=0}^{m} \int_\Omega b_i \nabla^i u \nabla^i v$$

on E;

(b) Sobolev's Inequality holds in E;

(c) E can be imbedded into $W^{m,2}_{\text{loc}}(\overline{\Omega})$.

Consequently, a bounded sequence of E must be bounded in $W^{m,2}(\Omega_r)$ for any $r > 0$.

2.5. Two Versions of the Mountain Pass Theorem.

As our problems have a variational structure and the norm $\|\cdot\|_\ell$ induced by the operator ℓ (see (2.4.2)) is an equivalent norm in $E(m)$, we will naturally choose a variational approach to deal with our problems. One of the most powerful tools to treat superlinear subcritical problems is the well-known Mountain Pass Theorem by Ambrosetti and Rabinowitz, see [Ra1] and references therein. We also employ the Mountain Pass Theorem to obtain the existence of solutions for our problems. In this section, for the convenience, we recall two versions of the Mountain Pass Theorem.

Suppose E is a real Banach space. Let $C^1(E,R)$ denote the set of functionals that are Fréchet differentiable and whose Fréchet derivatives are continuous on E. For $J \in C^1(E,R)$, we say J satisfies the Palais-Smale condition (henceforth denoted by (PS)) if any sequences $\{u_k\} \subset E$ for which $J(u_k)$ is bounded and $J'(u_k) \to 0$ as $k \to \infty$ possesses a convergent subsequence.

THEOREM A (MOUNTAIN PASS THEOREM). *Let E be a real Banach space and $J \in C^1(E, R)$ satisfying (PS) condition. Suppose $J(0) = 0$ and*

(a) there are constants ρ, $a > 0$ such that $J|_{\partial B_\rho} \geq a$,

and

(b) there is an $e \in E \backslash B_\rho$ such that $J(e) \leq 0$.

Then J possesses a critical value $c \geq a$. Moreover c can be characterized as

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} J(u) \tag{2.5.1}$$

where $\Gamma = \{g \in C([0,1], E) \mid g(0) = 0,\ g(1) = e\}$.

THEOREM B. *Let E be an infinite dimensional Banach space and let $J \in C^1(E, R)$ be even, satisfy (PS) condition, and $J(0) = 0$. If $E = V \oplus X$, where V is finite dimensional, and J satisfies*

(a) there are constants $\rho, a > 0$ such that $J|_{\partial B_\rho \cap X} \geq a$, and

(b) for each finite dimensional subspace $E_0 \subset E$, there is an $R_0 = R_0(E_0)$ such that $J \leq 0$ on $E_0 \backslash B_{R_0}$.

Then J possesses an unbounded sequence of critical values.

The proofs of the two theorems can be found on page 7 and page 56 of [Ral].

CHAPTER 3

SUBCRITICAL SECOND ORDER ELLIPTIC PROBLEMS

3.1. Introduction.

In this chapter we consider the existence of nontrivial positive solutions for the second order semilinear elliptic problems:

$$\begin{cases} \ell u = f(x,u) \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0, \quad \lim\limits_{|x|\to\infty} u = 0, \end{cases} \tag{I}$$

where Ω is a smooth unbounded domain in R^n, $n \geq 3$, ℓ is a uniformly elliptic operator

$$\ell u = -\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j u) + b(x)u$$

with $a_{ij} = a_{ji} \in C^{1+\alpha}_{\text{loc}}(\overline{\Omega}) \cap L^\infty(\overline{\Omega})$, $0 \leq b \in C^{\alpha}_{\text{loc}}(\Omega) \cap L^\infty(\overline{\Omega})$. The function $f(x,t)$ is assumed to have subcritical growth at ∞, i.e., $|f(x,t)| \leq C|t|^\alpha$ for t large, $0 < \alpha < \frac{n+2}{n-2}$. The specific conditions on f will be given in the following sections. Special attention is given to the superlinear case with prototype: $f(x,t) = g(x)t^\alpha$, $1 < \alpha < \frac{n+2}{n-2}$, though the sublinear case: $f(x,t) = h(x)t^\beta$, $0 < \beta < 1$ and the mixed sub-superlinear case: $f(x,t) = g(x)t^\alpha + h(x)t^\beta$ are also considered.

Problem (I) has been studied by several authors. We mention in particular: Atkinson and Peletier [AP], Berestycki and Lions [BL], Berger [Be], Coffman and Marcus [CM], Ding and Ni [DN1] [DN2], Fukagai [Fu1], Furusho [Fur], Gidas, Ni and Nirenberg [GNN], Gidas and Spruck [GS], Kawano [Ka], Kusano and Naito [KN], Kusano and Oharu [KO] and Ni [Ni1] [Ni2].

Most of these results were obtained either for the radially symmetric case: $\Omega = R^n$, $\ell = -\Delta + b$, b is a positive constant, $f(x,t) = f(|x|,t)$, or for the case in which $f(x,t)$ admitted suitable radial majorants. Their methods involved ODE theory, and the combination of variational calculus and symmetric rearrangement. There are fewer papers which deal with general nonradial cases for Ω unbounded (Note: If Ω is bounded, there are many references, see, e.g. [Ra1]), see, e.g., Chaljub-Simon and Volkmann [CV], Noussair and Swanson [NS1] [NS2] [NS3]. But the picture of the problem in the nonradial case is still not clear. Specifically, the prototype superlinear problem

$$\ell u = g(x)u^{\alpha}, \quad 1 < \alpha < \frac{n+2}{n-2}$$

was studied in the above references. In [NS1], Noussair and Swanson showed the existence of a positive solution by arguments involving the application of the Mountain Pass Theorem to a sequence of increasing bounded subdomains. In [CV], Chaljub-Simon and Volkmann used weighted Sobolev spaces. Although the conditions on f in these two papers are different, both assume $b(x) \geq b_0 > 0$. This is a significant assumption in the case of Ω unbounded. Noussair and Swanson considered the more general case of $b(x) \geq 0$ in [NS2] [NS3], and obtained the existence of a positive solution for $\lambda = \lambda_0$ (some $\lambda_0 > 0$) for the problem $\ell u = \lambda\ f(x,u)$, under the assumption

$$0 \leq f(x,u) \leq C(1+|x|^a)^{-1}|u|^{\alpha}$$

with $0 < a \leq 2$, $\frac{n+2-2a}{n-2} < \alpha < \frac{n+2}{n-2}$. A scaling argument gives a general result if f is homogeneous. Kawano, Sutsuma and Yotsutani [KSY] and Li and Ni [LN1] [LN2] considered the radial case: $\ell = -\Delta, g(x) = g(|x|)$, $\Omega = R^n$. The existence of a positive solution was obtained in [KSY] under the condition $\int_0^\infty g(r)r \; dr < \infty$, while in [LN1] under the condition $\int_0^\infty g(r)r^{\frac{n-\alpha(n-2)}{2}} dr < \infty$. In [LN2], the existence of a positive solution with finite total mass $\int_{R^n} g(|x|)u^\alpha dx < \infty$ was discussed.

For the sublinear problem

$$-\Delta u = h(|x|)u^\beta, \quad 0 < \beta < 1,$$

Fukagai [Fu1] obtained a positive solution if $\int_0^\infty h(r)r \; dr < \infty$.

For the mixed sub-superlinear problem

$$\ell u = g(x)u^\alpha + h(x)u^\beta, \quad 0 < \beta < 1 < \alpha,$$

Kusano and Trench [KT] studied the radial case, while Furusho [Fr] treated the non-radial case under restrictions on radial majorants of g and h. Allegretto and Huang [AH1] also studied the nonradial case and obtained the existence of solutions bounded above and below by positive constants.

We wish, in this chapter, to establish more general conditions under which (I) has a positive solution or infinitely many solutions. We are mainly interested in those problems having highly nonradial terms. Our tools are weighted spaces set up in Chapter 2 and various variants of the Mountain Pass Theorem. We consider the general case $b(x) \geq 0$ and do not require the existence of radial majorants. While our results thus cover the case with highly nonradial properties, we show that even in radial case, they are equivalent to earlier results in some sense. Actually, some examples show that our assumptions are optimal.

This chapter is organized as follows: after some preliminary discussion, we present our main results in Section 3.2. We prove, in particular, that solution of (I) will exist in the superlinear case if $|f(x,t)| \leq g(x)|t|^{\alpha}$ for $1 < \alpha < \frac{n+2}{n-2}$ with $g \in L^{\infty}(\Omega) \cap L^{p_0}(\Omega)$, some $p_0 = p_0(n, \alpha)$. In Section 3.3, we modify the method introduced in Section 3.2 in order to obtain existence criteria for sublinear problems and for mixed sup-superlinear problems. Then, in Section 3.4, we discuss the results obtained and give illustrative examples which explicitly compare our theorems to earlier work.

3.2. Superlinear Problems.

We consider the problem (I) in the space $E(1)(\equiv E)$, where E is defined in Chapter 2. Recall that E is the completion of $C_0^{\infty}(\Omega)$ under the norm

$$\|u\| = \left(\int_{\Omega} |\nabla u|^2 + w_0 u^2\right)^{\frac{1}{2}},$$

where $w_0 = \max\{\sigma(x),\ b(x)\}$. The norm induced by the operator ℓ

$$\|u\|_{\ell} = \left(\int_{\Omega} \sum a_{ij} D_i u D_j u + b u^2\right)^{\frac{1}{2}}$$

is equivalent to $\|\cdot\|$ by the uniform ellipticity of ℓ and Lemma 2.3 (d). Note that $E \sim W_0^{1,2}(\Omega)$ if $b(x) \geq b_0 > 0$, since $c_1 \leq w_0(x) \leq c_2$ for some constants $c_1, c_2 > 0$ and $x \in \Omega$. Also $u^+, u^- \in E$ for any $u \in E$, where $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$. For the proof, we refer to Section 7.4 and 7.5 of [GT], as the proof is local in nature, and E and $W_0^{1,2}(\Omega)$ have the same local properties. Again we note that the presence of the positive weight function $w_0(x)$ implies that $\|\cdot\|$ defines a norm on $W^{1,2}(\Omega_r)$ which is equivalent to the standard Sobolev norm. An immediate consequence of this fact is that any bounded sequence of E admits a convergent subsequence in $L^p(\Omega_r)$ for $1 \le p < \frac{2n}{n-2}$ even if $b \equiv 0$, by the Compact Embedding Theorem. This compactness property is used repeatedly in the sequel.

We state the hypotheses on f:

(1) f is locally Hölder continuous in $\overline{\Omega} \times R^+$, $0 < f(x,t)$ in $\Omega_0 \times R^+$ for some open set $\Omega_0 \subseteq \Omega$;

(2) $|f(x,t)| \le g(x)|t|^\alpha$ with $1 < \alpha < \frac{n+2}{n-2}$, $g \in L^\infty(\overline{\Omega}) \cap L^{p_0}(\Omega)$, $p_0 = \frac{2n}{2n-(\alpha+1)(n-2)}$;

(3) there exists $\mu > 2$ such that $\mu F(x,t) \le tf(x,t)$ for $(x,t) \in \Omega \times R^+$, where $F(x,t) = \int_0^t f(x,s)ds$.

In E we define

$$K(u) = \int_\Omega F(x,u)dx,$$
$$J(u) = \frac{1}{2}\|u\|_\ell^2 - K(u),$$

for $u \in E$. Under our assumption (2) on f, K and J are well defined by Sobolev's Inequality and $|K(u)| \le \int_\Omega g(x)|u|^{\alpha+1}dx \le \|g\|_{p_0}\|u\|^{\alpha+1}_{\frac{2n}{n-2}} \le C\|g\|_{p_0}\|u\|_\ell^{\alpha+1}$.

Lemma 3.1.

(a) K and J are weakly lower semicontinuous and differentiable in E with

$$K'(u)(\varphi) = \int_\Omega f(x,u)\varphi;$$

(b) K' is continuous and compact from E to E.

Proof: (a) Let $u_k \to u$ weakly in E. We select $\Omega_1 = \{x \in \Omega \mid \ |x| < r_1\}$ and observe:

$$\begin{aligned}&|K(u_k) - K(u)|\\ &\leq \int_{\Omega_1} |F(x,u_k) - F(x,u)| + C\|g\|_{L^{p_0}(\Omega\setminus\Omega_1)}(\|u_k\|_\ell^{\alpha+1} + \|u\|_\ell^{\alpha+1}).\end{aligned}$$

Since $\{u_k\}$ is bounded in E, hence $\{u_k|_{\Omega_1}\}$ is bounded in $W^{1,2}(\Omega_1)$ by Lemma 2.3 (b). It follows from the compact embedding $W^{1,2}(\Omega_1) \hookrightarrow L^p(\Omega_1)$ for $1 \leq p < \frac{2n}{n-2}$ (see, e.g., Theorem 6.2 of [Ad]) that there exists a subsequence $u_{k_j} \to u$ in $L^p(\Omega_1)$, whence $u_k \to u$ in $L^p(\Omega_1)$. Since $|F(x,t)| \leq \frac{1}{\alpha+1}g(x)|t|^{\alpha+1}$, Nemisky operator properties show $\int_{\Omega_1} F(x,u_k) \to \int_{\Omega_1} F(x,u)$. Therefore $K(u_k) \to K(u)$ since $g \in L^{p_0}(\Omega)$. The weak lower semicontinuity of J follows immediately.

For the differentiability of K, we show that: given any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, u) > 0$ such that

$$\left|\int_\Omega F(x,u+\varphi) - \int_\Omega F(x,u) - \int_\Omega f(x,u)\varphi\right| < \varepsilon\|\varphi\|_\ell$$

for all $\varphi \in E$ with $\|\varphi\|_\ell \leq \delta$. Observe that $g \in L^{p_0}(\Omega)$ and

$$\int_{\Omega\setminus\Omega_1} |F(x,u+\varphi) - F(x,u) - f(x,u)\varphi|$$

$$\le \int_{\Omega\setminus\Omega_1} g\{(|u|+|\varphi|)^\alpha|\varphi| + |u|^\alpha|\varphi|\}$$

$$\le C\|g\|_{L^{p_0}(\Omega\setminus\Omega_1)}(\|u\|_\ell^\alpha + \|\varphi\|_\ell^\alpha)\|\varphi\|_\ell$$

$$< \frac{\varepsilon}{2}\|\varphi\|_\ell$$

for sufficienty large r_1 and $\|\varphi\|_\ell \le 1$. To estimate the integral on the bounded domain:

$$\left|\int_{\Omega_1} F(x,u+\varphi) - F(x,u) - f(x,u)\varphi\right| < \frac{\varepsilon}{2}\|\varphi\|_\ell,$$

we need only follow the arguments in Proposition B10 of [Ra1].

(b) Let $\tilde{u}_k = K'(u_k)$, $\tilde{u} = K'(u)$. For continuity, it suffices to show that for any sequence $u_k \to u$ in E there exists a subsequence $\{u_{k_j}\}$ such that $\tilde{u}_{k_j} \to \tilde{u}$ in E. Note that

$$\begin{aligned} \|\tilde{u}_k - \tilde{u}\|_\ell \le C\{&\|f(\cdot,u_k) - f(\cdot,u)\|_{L^{2n/(n+2)}(\Omega_1)} + \\ &\|g\|_{L^{p_0}(\Omega\setminus\Omega_1)}(\|u_k\|_\ell^\alpha + \|u\|_\ell^\alpha)\}, \end{aligned} \tag{3.2.1}$$

and that $\{u_k\}$ has a subsequence $u_{k_j} \to u$ pointwise a.e. in Ω. The continuity of K' follows from $|f(x,t)|^{2n/(n+2)} \le C|t|^{2n\alpha/(n+2)}$ in Ω_1 with $1 < \frac{2n\alpha}{n+2} < \frac{2n}{n-2}$ and the continuity properties of the Nemisky operator. To show compactness, let $\Omega_i = \Omega \cap B_{r_j}$, $r_j \to \infty$ such that $\|g\|_{L^{p_0}(\Omega\setminus\Omega_j)} < \frac{1}{j}$. We observe that (3.2.1) is valid with Ω_1 replaced by Ω_j. For each j, the compactness of $W^{1,2}(\Omega_j) \hookrightarrow L^p(\Omega_j)$ for $1 \le p < \frac{2n}{n-2}$ and the boundedness of $\{u_k\}$ in $W^{1,2}(\Omega_j)$ imply the existence of a subsequence $\{u_k^j\}$ and function $v^j \in L^p \cap W^{1,2}(\Omega_j)$ such that $u_k^j \xrightarrow{k} v^j$ in $L^p(\Omega_j)$ and pointwise a.e. in Ω_j. Without loss of generality, we assume $v^j|_{\Omega_{j-1}} = v^{j-1} \equiv v$. Finally, note that $\{\tilde{u}_k^k\}$ is Cauchy in E. Indeed, if $\varepsilon > 0$ is given, choose j such that $\|g\|_{L^{p_0}(\Omega\setminus\Omega_j)} < \varepsilon$, and observe that for k, i large $\|f(\cdot, u_k^k) - f(\cdot, u_i^i)\|_{L^{2n/(n+2)}(\Omega_j)} < \varepsilon$ by the arguments used above. The compactness of K' follows immediately from an estimate similar to (3.2.1).

We know that the critical points of J, that is points u such that

$$J'(u)(\varphi) = \langle u, \varphi \rangle_\ell - \int_\Omega f(x,u)\varphi = 0 \tag{3.2.2}$$

for $\varphi \in E$, where $\langle \cdot, \cdot \rangle_\ell$ denotes the E inner product induced by ℓ (see the end of Chapter 2), are weak solutions of $\ell u = f(x,u)$. We observe the following properties of such solutions.

LEMMA 3.2. *Let u be a critical point of J.*

(a) If $f(x,t) \geq 0$ then $u \geq 0$;

(b) $u \in L^p(\Omega)$, $\frac{2n}{n-2} \leq p \leq \infty$. If $b(x) \geq b_0 > 0$ then $u \in L^p(\Omega)$, $2 \leq p \leq \infty$;

(c) $\lim\limits_{|x|\to\infty} u = 0$.

PROOF: (a) Setting $\varphi = u^-$ in (3.2.2) yields $\langle u^-, u^- \rangle_\ell = \int_\Omega f(x,u)u^- \leq 0$, and $u^- = 0$ follows.

(b) The proof is adapted from a procedure due to Brezis and Kato [BK] and is a variation of the method used by Figueiredo, Lions and Nussbaum [FLN]. We always assume $u \geq 0$ for otherwise we consider u^+ and u^- respectively. Let $u_k(x) = \min\{u(x), k\}$, $k = 1, 2, \ldots$. For any real number

$i \geq 1$, $(u_k)^i \in E$. It follows from (3.2.2) that:

$$\langle u, (u_k)^i \rangle \leq \int_\Omega |f(x,u)|(u_k)^i \leq \|g\|_\infty \int_\Omega u^{\alpha+i}.$$

But

$$\begin{aligned} \langle u, (u_k)^i \rangle &\geq \frac{4i}{(i+1)^2} \|\nabla (u_k)^{\frac{i+1}{2}}\|_2^2 \\ &\geq C(n,i)\Big(\|u_k\|_{\frac{n}{n-2}(i+1)}\Big)^{i+1} \end{aligned}$$

by Sobolev Inequality. Hence we have

$$\Big(\|u_k\|_{\frac{n}{n-2}(i+1)}\Big)^{i+1} \leq C(n,i,\|g\|_\infty)\int_\Omega u^{\alpha+i}.$$

Let $i = i_1 = 1+\sigma$, $p_1 = \frac{n}{n-2}(1+i_1) = \frac{2n}{n-2} + \frac{n}{n-2}\sigma$, where $\sigma = \frac{n+2}{n-2} - \alpha$. It follows that:

$$\|u_k\|_{p_1}^{i_1+1} \leq C\int_\Omega u^{\frac{2n}{n-2}} < \infty.$$

Therefore we have $u \in L^{p_1}(\Omega)$ by letting $k \to \infty$. This yields $u \in L^p(\Omega)$ for $\frac{2n}{n-2} \leq p \leq p_1$. Iterating this process gives:

$$\Big(\|u_k\|_{p_{j+1}}\Big)^{i_j+1} \leq C(n,i_j,\|g\|_\infty)\int_\Omega u^{p_j} < \infty$$

where $i_j = 1+\sigma+\left(\frac{n}{n-2}\right)\sigma+\cdots+\left(\frac{n}{n-2}\right)^j\sigma$, $\quad p_j = \frac{2n}{n-2}+\left(\frac{n}{n-2}\right)\sigma+\cdots+\left(\frac{n}{n-2}\right)^j\sigma$.

It follows from this and the assumptions that $f(x,u(x)) \in L^q(\Omega)$ for some $q > \frac{n}{2}$.

Set

$$\hat{f}(x,t) = \begin{cases} f(x,t) & x \in \Omega, \\ 0 & x \in R^n \backslash \Omega. \end{cases}$$

We apply Theorem 8.25 of [GT] and obtain:

$$|u(x)| \leq C\{\|u\|_{L^{2n/(n-2)}(B_1(x))} + \|\hat{f}(\cdot u)\|_{L^q(B_1(x))}\} \tag{3.2.3}$$

for some $q > \frac{n}{2}$. It follows that $u \in L^\infty(\Omega)$. Since $E \sim W_0^{1,2}(\Omega)$ if $b(x) \geq b_0 > 0$, this completes the proof by the standard embedding theorems.

(c) This is immediate from (3.2.3).

We now state our main existence criteria.

THEOREM 3.1. *Let f satisfy hypotheses (1) - (3). Then problem (I) has at least one positive decaying solution $u \in L^p(\Omega)$ for $\frac{2n}{n-2} \leq p \leq \infty$.*

PROOF: Without loss of generality, we set $f(x,t) = 0$ if $t \leq 0$. It suffices to verify that J satisfies the conditions of the Mountain Pass Theorem (see Theorem A of Section 2.5). Since

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|u\|_\ell^2 - \frac{1}{\alpha+1}\int_\Omega g(x)u^{\alpha+1} \\ &\geq \frac{1}{2}\|u\|_\ell^2 - C\|g\|_{L^{p_0}}\|u\|_\ell^{\alpha+1}, \end{aligned}$$

we conclude that there exist $a, \rho > 0$ such that $J(u) \geq a$ for all $u \in \partial B_\rho(0)$. From assumptions (1) and (3) we have: $0 < \mu F(x,t) \leq tf(x,t)$ for $(x,t) \in \Omega_0 \times R^+$. Without loss of generality, assume Ω_0 is bounded. Integrating shows that there exist $a_1, a_2 > 0$ such that

$$F(x,t) \geq a_1 t^\mu - a_2 \tag{3.2.4}$$

for $(x,t) \in \Omega_0 \times R^+$. Let $w \in C_0^\infty(\Omega_0)$ with $w(x) \geq 0$, $\not\equiv 0$, and let s be a positive number. We observe that

$$J(sw) \leq \frac{1}{2}s^2\|w\|_\ell^2 - s^\mu \int_\Omega a_1 w^\mu + a_2|\Omega_0|$$

yields $J(sw) < 0$ for s large. The compactness of $K'(\cdot)$ and (3) imply that the (PS) condition holds (see page 11 of [Ra1]). We conclude that $J(\cdot)$ has a nontrivial critical point, say u. From Lemma 3.2 it follows that $u \in L^p(\Omega)$ for $\frac{2n}{n-2} \leq p \leq \infty$ and $\lim\limits_{|x|\to\infty} u = 0$. Note that $f(x,t) = 0$ for $(x,t) \in \Omega \times (-\infty, 0]$ gives $u \geq 0$ a.e.. By $\ell u - \frac{f_-(x,u)}{u}u = f_+(x,u)$ and the Strong Maximum Principle (see Theorem 3.5 of [GT]) we have $u > 0$ in Ω.

If we add further conditions, we can show the existence of other solutions besides the positive one found above.

THEOREM 3.2. *Let f satisfy hypothesis (1) - (3) with $\Omega_0 = \Omega$ in (1), and f be odd in t. Then problem (I) has infinitely many nontrivial decaying solutions which are in $L^p(\Omega)$ for $\frac{2n}{n-2} \leq p \leq \infty$.*

PROOF: We apply Theorem B of Section 2.5. In view of Theorem 3.1, we need only verify that for each finite dimensional subspace $E_0 \subseteq E$ there exists $R = R(E_0)$ such that $J \leq 0$ on $E_0 \backslash B_R$. Let $\{e_i\}_{i=1}^k$ be an orthonormal basis for E_0. Each $v \in E_0$ has the form $v = \sum\limits_{i=1}^k \nu_i e_i$, $\nu \equiv (\nu_1, \ldots, \nu_k) \in R^k$. Denote $S_1 = \{v \in E_0 \mid |\nu| = 1\}$. For any $v \in S_1$, $0 < \int_\Omega |v|^\mu \leq \infty$ since $v(x) \not\equiv 0$. We claim that there are $\varepsilon_0, r_0 > 0$ such that

$$\int_{\Omega_{r_0}} |v|^\mu \geq \varepsilon_0 \tag{3.2.5}$$

for all $v \in S_1$. To see this, observe that $\|v\|_\ell = 1$ for $v \in S_1$, and, since E_0 is finite dimensional:

$$\|v\|_{L^2(\Omega_{r_0})} \sim \|v\|_{\ell(\Omega_{r_0})} \geq C$$

if r_0 is large enough, where $\|v\|_{\ell(\Omega_{r_0})} = (\int_{\Omega_{r_0}} \sum a_{i,j} D_i v D_j v + bv^2)^{1/2}$. Since $\mu > 2$, either $v \notin L^\mu(\Omega_{r_0})$ in which case (3.2.5) is immediate or we apply the Hölder Inequality:

$$\|v\|_{L^\mu(\Omega_{r_0})} \geq C\|v\|_{L^2(\Omega_{r_0})}.$$

There are $a_1, a_2 > 0$ such that

$$F(x,t) \geq a_1|t|^\mu - a_2 \tag{3.2.6}$$

for $(x,t) \in \Omega_{r_0} \times R$ by assumptions (1), (3) and f odd in t. Observing that any u in E_0 can be written as $u = Rv$, where $R = \|u\|_\ell$, some $v \in S_1$, and $F(x,t) \geq 0$, we have from (3.2.5) - (3.2.6)

$$\begin{aligned} J(u) &= \frac{1}{2}R^2 - \int_{\Omega_{r_0}} F(x,u) - \int_{\Omega\backslash\Omega_{r_0}} F(x,u) \\ &\leq \frac{1}{2}R^2 - \int_{\Omega_{r_0}} F(x,u) \leq \frac{1}{2}R^2 - a_1 R^\mu \int_{\Omega_{r_0}} |v|^\mu + a_2|\Omega_{r_0}| \\ &\leq \frac{1}{2}R^2 - a_1\varepsilon_0 R^\mu + a_2|\Omega_{r_0}|. \end{aligned}$$

This implies $J(u) < 0$ for all $u \in E_0\backslash B_R(0)$ for some R.

We remark that condition (3) could be weakened somewhat with the same proof. For example, if $g \in L^{\frac{n}{2}}(\Omega)$, then (3) is required only for large t. Alternatively, (3) could be replaced by

$$\begin{cases} \mu F(x,t) \leq tf(x,t) & x \in \Omega\backslash\Omega_1, \ t \geq 0, \\ \mu F(x,t) \leq tf(x,t) & x \in \Omega_1, \quad t \geq a, \end{cases}$$

for some $a > 0$, Ω_1 bounded. These can be seen from the estimate (3.2.7) below.

Furthermore, we remark that if $b(x) \geq b_0 > 0$ then some of the assumptions on $f(x,t)$ can be relaxed significantly. We next examine in more detail this case. Specifically, assume that the hypotheses now are:

(2′) $|f(x,t)| \leq g(x)\, |t|^\alpha$, $1 < \alpha < \frac{n+2}{n-2}$, $g \in L^\infty(\Omega)$, and $\lim\limits_{|x|\to\infty} \|g\|_{L^1(B_1(x))} = 0$;

(3′) There exist $\mu > 2$ and $a > 0$ such that

$$\mu F(x,t) \leq t f(x,t) \qquad \text{for} \quad (x,t) \in \Omega \times [a,\infty).$$

Observe that K and J in this case are well-defined by the embedding theorem: $\|u\|_p \leq C\|u\|_\ell$, $2 \leq p \leq \frac{2n}{n-2}$ for all $u \in E \sim \overset{\circ}{W}{}^{1,2}(\Omega)$.

LEMMA 3.3. *Let $b(x) \geq b_0 > 0$ and (2′) hold. Then*

(a) K and J are weakly lower semicontinuous and $K'(u)(\varphi) = \int_\Omega f(x,u)\varphi$.

(b) $K'(u)$ is continuous and compact from E to E.

PROOF: The arguments of the proof of Lemma 3.1 are basically still valid, except that $\|g\|_{L^{p_0}(\Omega\setminus\Omega_1)}$ and $\|g\|_{L^{p_0}(\Omega\setminus\Omega_j)}$ are replaced by $M_{\nu,1}(g,\Omega\setminus\Omega_1)$ and $M_{\nu,1}(g,\Omega\setminus\Omega_m)$, where

$$M_{\nu,p}(w,D) = \sup_{x\in D}\left[\int_{B_1(x)} |w(y)|^p |y-x|^{\nu-n} dy\right]$$

some $\nu : 0 < \nu < n - \frac{\alpha+1}{2}(n-2)$, $p \geq 1$. We employ the embedding theorem 2.3 of [BS] to show:

$$\begin{aligned}\int_{\Omega\backslash\Omega_1} g|u|^{\alpha+1} &= \left(\|g^{\frac{1}{\alpha+1}}u\|_{L^{\alpha+1}(\Omega\backslash\Omega_1)}\right)^{\alpha+1} \\ &\leq C\ M_{\nu,\alpha+1}(g^{\frac{1}{\alpha+1}},\ \Omega\backslash\Omega_1)\|u\|_\ell^{\alpha+1} \\ &= C\ M_{\alpha,1}(g,\Omega\backslash\Omega_1)\|u\|_\ell^{\alpha+1}.\end{aligned}$$

Observe that $M_{\nu,1}(g,\Omega\backslash\Omega_1)$ and $M_{\nu,1}(g,\Omega\backslash\Omega_j)$ can be made arbitrarily small by choosing Ω_1 and Ω_j sufficiently "large", since by $0 < \nu < n - \frac{\alpha+1}{2}(n-2)$, $g \in L^\infty$ and Hölder's Inequality:

$$\int_{|y-x|\leq 1} g(y)|y-x|^{\nu-n}dy \leq C(\|g\|_\infty, \nu)\|g\|_{L^1(B_1(x))}^b \to 0$$

as $|x| \to \infty$, for some $0 < b < \frac{\nu}{n}$.

THEOREM 3.3. *Let $b \geq b_0 > 0$ and f satisfy the hypothesis (1), (2′) and (3′). Then problem (I) has at least one decaying positive solution $u \in L^p(\Omega)$ for $2 \leq p \leq \infty$.*

PROOF: As in the proof of Theorem 3.1, assume $f(x,t) = 0$ for $t \leq 0$. We only show that $J(\cdot)$ satisfies the (PS) condition since the rest is the same. Let $\{u_k\} \subseteq E$ with $J(u_k) \leq d$ and $J'(u_k) \to 0$. We observe:

$$\begin{aligned}d &\geq \frac{1}{2}\|u_k\|_\ell^2 - \int_{u_k(x)\geq a} F(x,u_k) - \int_{0\leq u_k(x)<a} F(x,u_k) \\ &\geq \frac{1}{2}\|u_k\|_\ell^2 - \frac{1}{\mu}\int_{u_k(x)\geq a} f(x,u_k)u_k - \int_{0\leq u_k(x)<a} F(x,u_k)\end{aligned}$$

$$\geq \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|u_k\|_\ell^2 + \frac{1}{\mu}J'(u_k) - \int_{0\leq u_k(x)<a}\Big[F(x,u_k) - \frac{1}{\mu}f(x,u_k)u_k\Big].$$

We need only estimate the last term above:

$$\int_{0\leq u_k(x)<a}|F(x,u_k) - \frac{1}{\mu}f(x,u_k)u_k| \leq C\int_{0\leq u_k(x)<a} g|u_k|^{\alpha+1}$$

$$\leq C\Big\{\int_{\Omega_1\cap\{x|0\leq u_k(x)<a\}} gu_k^{\alpha+1} + \int_{(\Omega\backslash\Omega_1)\cap\{x|0\leq u_k(x)<a\}} gu_k^{\alpha+1} \tag{3.2.7}$$

$$\leq C\Big\{a^{\alpha+1}\|g\|_\infty|\Omega_1| + \int_{(\Omega\backslash\Omega_1)\cap\{x|0\leq u_k(x)<a\}} gu_k^{\alpha+1}\Big\},$$

but

$$\int_{(\Omega\backslash\Omega_1)\cap\{x|0\leq u_k(x)<a\}} gu_k^{\alpha+1} \leq a^{\alpha-1}\int_{\Omega\backslash\Omega_1} gu_k^2$$

$$\leq C\ a^{\alpha-1}M_{\nu,2}(g^{\frac{1}{2}},\Omega\backslash\Omega_1)\|u_k\|_\ell^2$$

$$\leq C\ a^{\alpha-1}M_{\nu,1}(g,\Omega\backslash\Omega_1)\|u_k\|_\ell^2,$$

$C = C(u,\alpha)$. So we obtain

$$d \geq \Big(\frac{1}{2} - \frac{1}{\mu} - C(n,\alpha,a)M_{\nu,1}(g,\Omega\backslash\Omega_1)\Big)\|u_k\|_\ell^2$$

$$+ \frac{1}{\mu}J'(u_k) - Ca^{\alpha+1}\|g\|_\infty|\Omega_1|.$$

Note that $M_{\nu,1}(g,\Omega-\Omega_1)$ can be very small if Ω_1 is sufficiently "large". It follows that $\{u_k\}$ is bounded, whence the (PS) condition holds.

REMARK: (a) In the case of $b(x) \geq b_0 > 0$, the analogue of Theorem 3.2 can be easily established. Under the additional condition: $\sum D_i a_{ij}$ is bounded, any solution u of (I) decays exponentially at ∞:

$$|u(x)| \leq C\ e^{-\delta|x|},$$

where $C,\ \delta > 0$, depending on u. We refer this to the proof of Theorem 4.7 of [NS1].

(b) With slight modifications, we can treat the more general case:

$$|f(x,t)| \leq \sum_{i=0}^{N} g_i(x)|t|^{\alpha_i},$$

for $\alpha_0 = 0,\ 1 < \alpha_1 < \cdots < \alpha_N < \frac{n+2}{n-2}$, provided that $g_0 \in L^2_{w_0^{-1}}(\Omega),\ g_i \in L^{p_i}(\Omega),\ p_i = \frac{2n}{2n-(\alpha_i+1)(n-2)},\ 1 \leq i \leq N$, and $\lim\limits_{t\to 0+} \frac{f(x,t)}{w_0 t} = 0$ uniformly in $x \in \Omega$ if $g_0(x) \not\equiv 0$.

3.3. Other Problems.

We now proceed to study the sublinear problem and the mixed sub-superlinear problem in the nonradial case.

For the sublinear problem we impose the conditions:

(4) $0 \leq f(x,t) \leq h(x)t^{\beta},\ t \geq 0,\quad 0 \leq \beta < 1,\quad h \in L^{\infty}(\Omega) \cap L^{q_0}(\Omega),\quad q_0 = \frac{2n}{2n-(\beta+1)(n-2)}$.

(5) $f_+(x,t) \geq h_0(x)t^{\beta_0}$ as $t \to 0^+$, where $h_0(x) \geq 0,\ \not\equiv 0,\ 0 \leq \beta_0 \leq \beta$.

THEOREM 3.4. *Let f satisfy (1), (4) and (5). Then problem (I) has a positive decaying solution.*

PROOF: We will assume $f(x,t) = f(x,0)$ for $t \leq 0$. By condition (4) and the proof of Lemma 3.1, K and J are still weakly lower semicontinuous. By

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|u\|_{\ell}^2 - \frac{1}{\beta+1}\int h(x)u^{\beta+1}dx \\ &\geq \frac{1}{2}\|u\|_{\ell}^2 - \frac{C}{\beta+1}\|h\|_{q_0}\|u\|_{\ell}^{\beta+1}, \end{aligned}$$

J is bounded below. Hence (I) has a solution u:

$$J(u) = \inf\{J(v) \mid v \in E\}.$$

We note that u must be a nontrivial solution since

$$J(s\varphi) \leq \frac{s^2}{2}\|\varphi\|_\ell^2 - \frac{s^{\beta+1}}{\beta+1}\int_\Omega h_0(x)|\varphi|^{\beta+1} < 0$$

for some $\varphi \in C_0^\infty(\Omega)$ and small $s > 0$. The decay of u follows from the proof of Lemma 3.2 without any modifications.

For the mixed sub-superlinear problem we have the following theorem.

THEOREM 3.5. *Let $f(x,t) = f_1(x,t) + f_2(x,t)$, let f_1 satisfy (1) - (3) and f_2 satisfy (1) and (4). Then problem (I) has a positive decaying solution, provided that*

$$2A_1^2\|g\|_{p_0}^{\frac{1-\beta}{\alpha-\beta}}\|h\|_{q_0}^{\frac{\alpha-1}{\alpha-\beta}}\left[\frac{1}{\alpha+1}\left(\frac{(\alpha+1)(1-\beta)}{(\beta+1)(\alpha-1)}\right)^{\frac{\alpha-1}{\alpha-\beta}}+\frac{1}{\beta+1}\left(\frac{(\beta+1)(\alpha-1)}{(\alpha+1)(1-\beta)}\right)^{\frac{1-\beta}{\alpha-\beta}}\right] < 1, \tag{3.3.1}$$

where A_1 is the Sobolev embedding constant (see [Ma]):

$$A_1 = \frac{(n-1)}{n\sqrt[n]{v_n}}\,\frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n}{2})}, \qquad v_n = \text{volume of } B_1(0).$$

PROOF: We employ the Mountain Pass arguments to obtain the existence of a nontrivial critical point of J. Here we assume $f(x,t) = f(x,0)$ for $t \leq 0$. In this case,

$$K(u) = \int_\Omega F_1(x,u) + \int_\Omega F_2(x,u),$$
$$J(u) = \frac{1}{2}\|u\|_\ell^2 - \int_\Omega (F_1(x,u) + F_2(x,u))$$

where $F_1(x,u) = \int_0^u f_1(x,s)ds$, $F_2(x,u) = \int_0^u f_2(x,s)ds$. J is differentiable, while K' is compact. Observe that for some $(0 \leq) w \in C_0^\infty(\Omega_0)$,

$$J(sw) \leq \frac{1}{2}s^2\|w\|_\ell^2 - s^\mu \int_\Omega a_1 w^\mu + a_2|\Omega_0|$$

for large $s > 0$, and the estimate

$$\begin{aligned}
J(u) &= \frac{1}{2}\|u\|_\ell^2 - \int_\Omega F_1(x,u) - \int_\Omega F_2(x,u) \\
&= \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|u\|_\ell^2 + \frac{1}{\mu}J'(u)(u) + \int_\Omega \Big(\frac{1}{\mu}f_1(x,u)u - F_1(x,u)\Big) \\
&\qquad + \int_\Omega \Big(\frac{1}{\mu}f_2(x,u)u - F_2(x,u)\Big) \\
&\geq \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|u\|_\ell^2 + \frac{1}{\mu}J'(u)(u) - C\Big(1 + \frac{1}{\mu}\Big)\|h\|_{q_0}\|u\|_\ell^{\beta+1}.
\end{aligned}$$

It follows that any sequence $\{u_k\}$ such that $J(u_k) \leq d$ and $J'(u_k) \to 0$ is bounded. Thus $\{u_k\}$ contains a convergent subsequence by the compactness of K' and $J'(u_k) \to 0$, and the (PS) condition follows. But the step $J(u) \geq a$ for $u \in \partial B_\rho(0)$ no longer follows as before. However:

$$\begin{aligned}
J(u)\big|_{\|u\|_\ell=\rho} &\geq \Big(\frac{1}{2}\|u\|_\ell^2 - \frac{1}{\alpha+1}A_1^{\alpha+1}\|g\|_{p_0}\|u\|_\ell^{\alpha+1} \\
&\qquad - \frac{1}{\beta+1}A_1^{\beta+1}\|h\|_{q_0}\|u\|_\ell^{\beta+1}\Big)\Big|_{\|u\|_\ell=\rho} \\
&= \frac{1}{2}\rho^2\Big(1 - \frac{2}{\alpha+1}A_1^{\alpha+1}\|g\|_{p_0}\rho^{\alpha-1} - \frac{2}{\beta+1}A^{\beta+1}\|h\|_{q_0}\rho^{\beta-1}\Big) \\
&\equiv \frac{1}{2}\rho^2 H(\rho).
\end{aligned}$$

Elementary differentiation shows that $H(\rho)$ has absolute maximum number $\rho_0 = \frac{1}{A_1}\left[\frac{(\alpha+1)(1-\beta)\|h\|_{q_0}}{(\beta+1)(\alpha-1)\|g\|_{p_0}}\right]^{\frac{1}{\alpha-\beta}}$. By the assumption (3.3.1), at ρ_0

$$H(\rho_0)$$

$$= 1 - 2A_1^2\|g\|_{p_0}^{\frac{1-\beta}{\alpha-\beta}}\|h\|_{q_0}^{\frac{\alpha-1}{\alpha-\beta}}\left[\frac{1}{\alpha+1}\left(\frac{(\alpha+1)(1-\beta)}{(\beta+1)(\alpha-1)}\right)^{\frac{\alpha-1}{\alpha-\beta}} + \frac{1}{\beta+1}\left(\frac{(\beta+1)(\alpha-1)}{(\alpha+1)(1-\beta)}\right)^{\frac{1-\beta}{\alpha-\beta}}\right]$$

$$> 0.$$

Hence $J(u)|_{\|u\|_\ell} = \rho_0 > 0$. By the Mountain Pass Theorem, $J(\cdot)$ has a nontrivial critical point, say u. A slightly modified proof of Lemma 3.2 shows the decay of u: in this case the estimate of the proof of Lemma 3.2 is: for $i \geq 1$,

$$\begin{aligned}
&\int_\Omega f(x,u)u^i = \int_\Omega f_1(x,u)u^i + \int_\Omega f_2(x,u)u^i \\
&\leq \|g\|_\infty \int_\Omega u^{\alpha+i} + \int_{0\leq u\leq 1} hu^{\beta+i} + \int_{1\leq u} hu^{\beta+i} \\
&\leq (\|g\|_\infty + \|h\|_\infty)\int_\Omega u^{\alpha+i} + \int_\Omega hu^{\beta+1} \\
&\leq (\|g\|_\infty + \|h\|_\infty)\int_\Omega u^{\alpha+i} + C\|h\|_{q_0}\|u\|_\ell^{\beta+1}.
\end{aligned}$$

The rest is the same. This completes the proof.

Note that if $f_2(x,t) \equiv 0$, then we recover the superlinear result of Theorem 3.1.

3.4. Comparison and Examples.

We conclude this chapter with illustrative examples which compare our results to earlier ones. We recall that we do not require $f(x,t)$ to be bounded by suitable radial majorants nor $b(x) \geq b_0 > 0$. Even in radial cases, the criterion of Theorem 3.1 is the best for superlinear problem in some sense.

Example 1. Consider the following typical superlinear problem.

$$\begin{cases} -\Delta u = f(x,u), & x \in R^n \\ \lim\limits_{|x| \to \infty} u = 0, & \end{cases} \tag{3.4.1}$$

and eigenvalue problem

$$\begin{cases} -\Delta u = \lambda f(x,u), & x \in R^n \\ \lim\limits_{|x| \to \infty} u = 0, & \end{cases} \tag{3.4.2}$$

where $n > 2$.

Suppose $f(x,t) \sim g(x)t^\alpha$, $0 < g(x) = 0(|x|^{-a})$ at ∞, $0 < a$, $\max\left\{\frac{n+2-2a}{n-2}, 1\right\} < \alpha < \frac{n+2}{n-2}$; Observe that $g \in L^{p_0}(R^n)$, $p_0 = \frac{2n}{2n-(\alpha+1)(n-2)}$ since $ap_0 > n$. Thus f satisfies (1) - (3), and by Theorem 3.1, problem (3.4.1) has a positive solution u. Equally (3.4.2) has a positive solution $u_\lambda(x)$ for any $\lambda > 0$.

Problem (3.4.2) has been considered by Noussair and Swanson [NS2] and [NS3] under the same conditions. Only for $\lambda = \lambda_0$ (some λ_0, not any), the existence of a positive solution $u_{\lambda_0}(x)$ is obtained.

Example 2. We still consider problem (3.4.1). Let $f(x,t) \cong g(|x|)t^\alpha$, $1 < \alpha < \frac{n+2}{n-2}$, $0 < g(|x|) = 0(|x|^{-a})$, $a > 0$. Problem (3.4.1) has a positive solution by Theorem 3.1 if $\frac{2na}{2n-(\alpha+1)(n-2)} > n$, i.e.

$$a > \frac{n-2}{2}\left[\frac{n+2}{n-2} - \alpha\right]. \tag{3.4.3}$$

The graph of $a_*(\alpha) \equiv \frac{n-2}{2}\left[\frac{n+2}{n-2} - \alpha\right]$ is a straight line with $2 = a_*(1) > a_*(\alpha) > a_*\left(\frac{n+2}{n-2}\right) = 0$, $1 < \alpha < \frac{n+2}{n-2}$.

Kaswano, Satsuma and Yotsutani [KSY] investigated problem (3.4.1) in this case. The existence of a positive solution is guaranteed there by the assumption $\int_0^\infty g(r)r\ dr < \infty$. If $g(|x|) = 0(|x|^{-a})$, this integration condition requires $a > 2$, which is much stronger than (3.4.3).

Li and Ni [LN1] also considered (3.4.1) in the radial case $f(x,u) = g(|x|)u^\alpha$. A radial positive solution is obtained there under the condition $\int_0^\infty g(r)r^{(n-\alpha(n-2))/2}dr < \infty$, which, when $g(r) = 0(r^{-a})$, requires a to satisfy (3.4.3). So their condition coincides with ours in this case.

The condition $g \in L^{p_0}$, $p_0 = \frac{2n}{2n-(\alpha+1)(n-2)}$ may be the best condition on g for superlinear problem, as shown by Kusano and Naito [KN]: the equation

$$-\Delta u = \frac{1}{1+|x|^a}u^\alpha$$

has no positive radial solutions in $C^2(R^n)$ for $a < \frac{n-2}{2}\left[\frac{n+2}{n-2} - \alpha\right]$, $1 < \alpha < \frac{n+2}{n-2}$.

Example 9. Consider the superlinear problem,

$$\begin{cases} -\Delta u + bu = f(x,u), & x \in R^n \\ \lim\limits_{|x|\to\infty} u = 0, \end{cases} \tag{3.4.4}$$

where $n > 2$, b is a positive constant.

Suppose $f(x,t) = g(x)t|t|^{\alpha-1}$, $1 < \alpha < \frac{n+2}{n-2}$, $g(x) > 0$, $x \in R^n$, and $\|g\|_{L^1(B_1(x))} \to 0$ as $|x| \to \infty$. Then problem (3.4.4) has infinitely many solutions, at least one of which is positive, such that $|u(x)| \leq C\ e^{-\delta|x|}$, some C, $\delta > 0$ depending on u. By the result of Noussair and Swanson [NS1], only one positive solution can be obtained in this case. Chaljub-Simon and Volkmann [CSV] considered (3.4.4) using a different method under the condition

$g(x) \geq C\ e^{-\delta|x|^2}$ at ∞, for some $C,\ \delta > 0$ and obtained the existence of a single solution.

We still consider (3.4.4) but suppose $b = 0$, $f(x,t) = g(x)t|t|^{\alpha-1}$, $1 < \alpha < \frac{n+2}{n-2}$, $0 < g(x) \in L^{p_0}$, and g is nonradial. The existence of infinitely many solutions (at least one is positive) is guaranteed by Theorem 3.1 and 3.2. To the best of our knowledge, this can not be obtained by any earlier results.

Example 4. Consider the mixed sub-superlinear problem

$$\begin{cases} -\Delta u = \lambda(g(x)u^\alpha + h(x)u^\beta), & x \in R^n \\ \lim\limits_{|x|\to\infty} u = 0, \end{cases} \tag{3.4.5}$$

where $n > 2$, $\lambda > 0$, $1 < \alpha < \frac{n+2}{n-2}$, $g \geq 0$, $\not\equiv 0$, $g \in L^{p_0}(R^n)$, $P_0 = \frac{2n}{2n-(\alpha+1)(n-2)}$, $0 < \beta < 1$, $0 \leq h \in L^{q_0}(R^n)$, $q_0 = \frac{2n}{2n-(\beta+1)(n-2)}$.

There exists a $\lambda^* > 0$ such that for $0 < \lambda < \lambda^*$, (3.3.1) of Theorem 3.5 is satisfied. Thus (3.4.5) has a positive solution for $0 < \lambda < \lambda^*$.

Kusano and Trench [KT] studied the radial case of (3.4.5) when $\lambda = 1$, while Furusho [Fur] considered this case under restrictions on the radial majorants of g and h. Their conditions are not comparable to the conditions of Theorem 3.5 even in the radial case.

Finally we note that the sublinear problem

$$\begin{cases} -\Delta u = h(x)u^\beta, & x \in R^n \\ \lim\limits_{|x|\to\infty} u = 0, \end{cases}$$

$0 < \beta < 1$, was considered by Fukagai in [Fu1], where an entire positive solution was obtained for $|h(x)| \leq h^*(|x|)$ and $\int_0^\infty h^*(r)r\ dr < \infty$. In the radial case, condition $\int_1^\infty h^*(r)r\ dr < \infty$ is weaker than $h \in L^{q_0}$, $q_0 = \frac{2n}{2n-(\beta+1)(n-2)}$ of

Theorem 3.4. However, in the cases where radial majorants are not available, our result is applicable but the radial argument test fails.

CHAPTER 4

HIGHER ORDER ELLIPTIC PROBLEMS

4.1. Introduction.

We extend the method introduced in Chapter 3 to study higher order elliptic problems. Specifically we are concerned with the existence of nontrivial decaying solutions of $2m^{\text{th}}$ order elliptic problem

$$\begin{cases} \ell u = f(x,u), & x \in \Omega \\ D^\nu u|_{\partial\Omega} = 0, & 0 \le |\nu| \le m-1 \\ \lim\limits_{|x|\to\infty} u = 0, & \end{cases} \tag{II}$$

where $\ell = \prod_{i=1}^{m}(-\Delta + b_i') \overset{\text{equivalently}}{=} \sum_{i=0}^{m} b_i(-\Delta)^i$, $b_m = 1$, b_i and $b_i' \in R^+$, Ω is an unbounded subdomain of R^n, $n > 2m$, ν is a non-negative multi-integer. Of special interest to us is the case of $\ell = (-\Delta)^m$ and $f(x,t)$ is purely superlinear with subcritical growth:

$$(-\Delta)^m u = g(x)u^\alpha,$$

$1 < \alpha < \frac{n+2m}{n-2m}$. As we did in Chapter 3, we also consider sublinear or mixed sub-superlinear problems.

Unlike second order elliptic problems, there are relatively few studies of higher order elliptic problems. The first papers on higher order elliptic problems are due to Walter [Wa1], [Wa2], and Walter and Rhee [WR], where $\Delta^m u = e^u$ and $\Delta^m u = f(|x|, u)$, in R^n, were studied. Higher order elliptic problems have received more attention since 1987. Dalmasso [Da1] [Da2], and Kusano and Swanson [KS] studied the biharmonic problem $\Delta^2 u = f(|x|, u)$ in R^n. Fukagai

[Fu2] studied semilinear problem of $\prod_{i=1}^{m}(-\Delta + b_i')u = f(|x|, u)$ in R^n. Edelson [Ed], Kusano, Naito and Swanson [KNS1] [KNS2] [KNS3] considered $(-\Delta)^m u = f(|x|, u)$. Also Kusano, Naito and Swanson [KNS4], and Usami [Us] dealt with the quasilinear problem $(-\Delta)^m u = f(|x|,\ u,\ \Delta u, \ldots, \Delta^{m-1}u)$ in R^n. Allegretto and Huang [AH2] studied more general problem $\ell_1 \cdots \ell_m \vec{u} = f(x, \vec{u}, \nabla \vec{u})$, where ℓ_i is second order elliptic operator. Bernis [Ber] considered $(-\Delta)^m u + |u|^{p-1}u = f(x)$. All of these papers employed radially symmetric arguments except [AH2] and [Ber]. The results obtained in these references can be divided into three types:

(a) The existence of positive unbounded solutions with special behaviour as $|x| \to \infty$, for instance, the solution $u(x) \sim |x|^{2m-i}$ at ∞, $i = 1, \ldots, m$;

(b) The existence of positive solutions bounded above and below by positive constants;

(c) The existence of positive decaying solutions in very restrictive cases.

As the objective of this chapter is (c) type existence, we mention further related papers and their results. Kusano, Naito and Swanson [KNS2] obtained a positive decaying solution for the sublinear and singular problem:

$$(-\Delta)^m u = h(|x|)u^\beta$$

$-1 < \beta < 1$ under the assumption

$$\int_0^\infty h(r) r^{N-1-\beta(N-2m)} dr < \infty, \tag{4.1.1}$$

while for the mixed sub-superlinear problem:

$$(-\Delta)^m u = g(|x|)u^\alpha + h(|x|)u^\beta$$

$0 < \beta < 1 < \alpha$ they postulated the further assumption (in addition to (4.1.1)):

$$\int_0^\infty g(r) r^{N-1-\alpha(N-2m)} dr < \infty \tag{4.1.2}$$

and

$$[I_2(g\rho^\alpha)]^{\frac{1-\beta}{\alpha-\beta}} [I_2(h\rho^\beta)]^{\frac{\alpha-1}{\alpha-\beta}} \left[\left(\frac{\alpha-1}{1-\beta} \right)^{\frac{1-\beta}{\alpha-\beta}} + \left(\frac{1-\beta}{\alpha-1} \right)^{\frac{\alpha-1}{\alpha-\beta}} \right] \leq 1 \tag{4.1.3}$$

(See Page 1299 of [KNS2] for the definitions of I_2 and ρ). The existence of positive decaying solution for superlinear problem was given as an open question in this paper. Dalmasso [Dal] partially answered this open question, and obtained a positive decaying solution of

$$\Delta^2 u = g(|x|)u^\alpha$$

$1 < \alpha < \frac{n+4}{n-4}$ by the assumption

$$\int_0^\infty g(r) r^3 dr < \infty. \tag{4.1.4}$$

Basically the existence of decaying solutions of superlinear problem remains open. We will give an answer to this open question by establishing general nonradial conditions for the superlinear problem of (II) to have a positive decaying solution. Moreover, we will treat sublinear and mixed sub-superlinear problems. Our conditions, as shown by some examples, are better than the former ones, such as (4.1.1) and (4.1.4), not only in the general case but also in some radial cases.

Our main tools are again weighted spaces and the Mountain Pass arguments. To show the decay of the solutions, we employ Agmon's L^p regularity results as the growth estimates, due to the failure of the iteration technique used in the proof of Lemma 3.2. The general philosophy follows the one given in Chapter 3 although the details are quite different. We conclude the chapter with some illustrative examples explicitly comparing our results to earlier ones.

4.2. Superlinear Problems.

In this section we derive existence theorem for the superlinear case of (II). We choose $E(m)$ as our function space, and, for convenience we denote $E(m)$ by E. Recall the following properties:

(a) $\|u\|_\ell^2 = \langle u, u\rangle_\ell = \sum_{i=0}^{m} \int_\Omega b_i |\Delta^i u|^2 dx$ is an equivalent norm in E;

(b) Sobolev's Inequality: $\|u\|_{\frac{2n}{n-2m}} \leq C\|u\|_\ell$ holds in E;

(c) E can be embedded into $W_{\text{loc}}^{m,2}(\overline{\Omega})$: $\|u\|_{W^{m,2}(\Omega_r)} \leq C\|u\|_\ell$.

We now state our hypotheses on f:

(1) $f \in C_{\text{loc}}^{\alpha}(\Omega \times R^+)$, $0 \leq f(x,t)$ in $\Omega \times R^+$ and $0 < f(x,t)$ in $\Omega_0 \times R^+$ for some open $\Omega_0 \subseteq \Omega$;

(2) $|f(x,t)| \leq g(x)|t|^\alpha$, $1 < \alpha < \frac{n+2m}{n-2m}$, $g \in L^\infty(\Omega) \cap L^{p_0}(\Omega)$, $p_0 = \frac{2n}{2n-(\alpha+1)(n-2m)}$;

(3) There exists $\mu > 2$ such that $\mu F(x,t) \leq tf(x,t)$ for $(x,t) \in \Omega \times R^+$ where $F(x,t) = \int_0^t f(x,s)ds$.

In E we define two functions:

$$K(u) = \int_{\Omega} F(x, u)dx,$$
$$J(u) = \frac{1}{2}\|u\|_{\ell}^{2} - K(u),$$

for any $u \in E$. Clearly K and J are well defined in E by assumption (2) and Sobolev's Inequality.

LEMMA 4.1.

(a) K and J are weakly lower semicontinuous and differentiable in E with $K'(u)(\varphi) = \int_{\Omega} f(x, u)\varphi dx$;

(b) K' is continuous and compact from E to E.

PROOF: Since the proof follows the lines of the one given in Section 3.2 for $m = 1$, we only sketch the basic ideas.

(a) Let $u_k \to u$ weakly in E. Then $\{u_k\}$ is bounded in E and we observe:

$$\begin{aligned}&|K(u_k) - K(u)|\\ &\leq \int_{\Omega_r} |F(x, u_k) - F(x, u)| + C\|g\|_{L^{p_0}(\Omega\setminus\Omega_r)}\left(\|u_k\|_{\ell}^{\alpha+1} + \|u\|_{\ell}^{\alpha+1}\right).\end{aligned}$$

The weak lower semicontinuity of K now follows from the boundedness of $\{\, u_k|_{\Omega_r} \,\}$ in $W^{m,2}(\Omega_r)$ (see Lemma 2.3 (b)) and the compactness of $W^{m,2}(\Omega_r) \hookrightarrow L^p(\Omega_r)$ for $1 \leq p < \frac{2n}{n-2m}$. The weak lower semicontinuity of J is then obvious.

For the differentiability of K, we show that: given any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, u) > 0$ such that

$$\left|\int_{\Omega} F(x, u + \varphi) - F(x, u) - f(x, u)\varphi\right| < \varepsilon\|\varphi\|_{\ell}$$

for any $\varphi \in E$ with $\|\varphi\|_\ell < \delta$. Observe that $g \in L^{p_0}(\Omega)$ and

$$\begin{aligned}&\int_\Omega F(x,u+\varphi) - F(x,u) - f(x,u)\varphi| \\ &\leq |\int_{\Omega_r} F(x,u+\varphi) - F(x,u) - f(x,u)\varphi| \\ &+ C\|g\|_{L^{p_0}(\Omega\setminus\Omega_r)}(\|u\|_\ell^\alpha + \|\varphi\|_\ell^\alpha)\|\varphi\|_\ell.\end{aligned}$$

We need only follow now the arguments of [Ra1, Prop. B10] to obtain the desired estimate for the integral on Ω_r.

(b) The proof of the continuity of K' is similar to the proof of the weak lower continuity of K, the only difference is that the estimate in this case is:

$$\begin{aligned}&\|K'(u_k) - K'(u)\| \\ &\leq C\{\|f(\cdot,u_k) - f(\cdot,u)\|_{L^{2n/(n+2m)}(\Omega_r)} + \|g\|_{L^{p_0}(\Omega\setminus\Omega_r)}(\|u_k\|_\ell^\alpha + \|u\|_\ell^\alpha)\}.\end{aligned}$$

To show compactness, note that:

$$\begin{aligned}K'(u)(\varphi) &= \int_{\Omega_r} f(x,u)\varphi + \int_{\Omega\setminus\Omega_r} f(x,u)\varphi \\ &\equiv K_r'(u)(\varphi) + R_r(u)\varphi.\end{aligned}$$

$K_r'(\cdot)$ is compact from E to E. Indeed, any bounded sequence $\{u_k\}$ in E is also bounded in $W^{m,2}(\Omega_r)$, whence it has a Cauchy subsequence in $L^p(\Omega_r)$ for $1 \leq p < \frac{2n}{n-2m}$. The compactness is immediate from the estimate below:

$$\|K_r'(u_k) - K_r'(u)\| \leq C\|f(\cdot,u_k) - f(\cdot,u)\|_{L^{2n/(n+2m)}(\Omega_r)}.$$

While $\|R_r(u)\| \leq C\|g\|_{L^{p_0}(\Omega\setminus\Omega_r)}\|u\|_\ell^\alpha$ implies that $K'(u)$ is a limit map of a sequence of compact maps under the norm $\|\cdot\|_\ell$. Therefore K' is also compact.

It is clear that a critical point of J, i.e.

$$J'(u)(\varphi) = \langle u, \varphi\rangle_\ell - \int_\Omega f(x,u)\varphi = 0$$

is a weak solution of $\ell u = f(x,u)$. The following lemma guarantees that critical points of J are classical solutions of $\ell u = f(x,u)$, and that their derivatives up to $2m-1$ decay at ∞.

LEMMA 4.2. *Let u be a critical point of J.*

(a) u is a classical solution of $\ell u = f(x,u)$ and furthermore, for any $p \geq \frac{2n}{n-2m}$,

$$\|u\|_{W^{2m,p}(\Omega\cap B_1(x))} \leq H\Big(\|u\|_{L^{\frac{2n}{n-2m}}(\Omega\cap B_2(x))}\Big),$$

where $H(\cdot)$ is a continuous function, dependent on n, m and p, with $H(0) = 0$,

(b) $D^\nu u \to 0$ as $|x| \to \infty$ for $|\nu| \leq 2m-1$.

PROOF: Suppose u is a critical point of J, i.e.,

$$\langle u, \varphi\rangle_\ell = \int_\Omega f(x,u)\varphi, \quad \varphi \in E.$$

It follows that for $\varphi \in C_0^\infty(\Omega \cap B_2(x))$,

$$\Big|\int_{\Omega\cap B_2(x)} u(\ell\varphi)\Big| = \Big|\int_{\Omega\cap B_2(x)} f(\cdot,u)\varphi\Big|$$

$$\leq \|g\|_\infty \int_{\Omega\cap B_2(x)} |u|^\alpha|\varphi|$$

$$\le C\|u\|^{\alpha}_{\frac{2n}{n-2m},\Omega\cap B_2(x)}\|\varphi\|_{p_1',\Omega\cap B_2(x)},$$

where $\frac{1}{p_1'}+\frac{1}{p_1}=1$, $p_1=\frac{q_1}{\alpha}$, $q_1=\frac{2n}{n-2m}$. We conclude from Agmon's regularity theorem 6.1 and 6.2 [Ag] that:

$$u\in W^{2m,p_1}(\Omega\cap B_{r_1}(x)),\quad 1<r_1<2$$

and

$$\|u\|_{W^{2m,p_1}(\Omega\cap B_{r_1}(x))}\le C\Big(\|u\|_{p_1,\Omega\cap B_2(x)}+\|u\|^{\alpha}_{\frac{2n}{n-2m},\Omega\cap B_2(x)}\Big)$$

$$\le C(\|u\|_{\frac{2n}{n-2m},\Omega\cap B_2(x)}+\|u\|^{\alpha}_{\frac{2n}{n-2m},\Omega\cap B_2(x)}),$$

where $C=C(n,m,\|g\|_{\infty},r_1)$. We obtain by the Embedding Theorem that:

$$u\in L^{q_2}(\Omega\cap B_{r_1}(x)),\quad q_2=\frac{2n}{\alpha(n-2m)-4m},$$

and

$$\|u\|_{q_2,\Omega\cap B_{r_1}(x)}\le C(\|u\|_{\frac{2n}{n-2m},\Omega\cap B_2(x)}+\|u\|^{\alpha}_{\frac{2n}{n-2m},\Omega\cap B_2(x)}),$$

unless $\alpha(n-2m)-4m$ is nonpositive, in which case clearly $u\in L^{q}(\Omega\cap B_{r_1}(x))$ for q large. Starting with q_2 and $p_2=\frac{q_2}{\alpha}$, iterating the process above i times yield:

$$u\in W^{2m,p_i}(\Omega\cap B_{r_i}(x)),\quad 1<r_i<\cdots<r_1<2,$$

$$p_i=\frac{q_i}{\alpha},\quad q_i=\frac{2n}{\alpha[\cdots\alpha[\alpha(n-2m)-4m]\cdots]-4m}.$$

After a finite number of steps, the denominator of q_i must be nonpositive and the result follows. The function $H(\cdot)$ is constructed merely by keeping track of the bound in each iteration step.

It follows from (a) that $D^\nu u \in W^{2m-|\nu|,p}(\Omega \cap B_1(x))$ for $|\nu| \leq 2m-1$. (b) is an immediate consequence of (a), the Embedding Theorem on $\Omega \cap B_1(x)$ and $\lim\limits_{|x|\to\infty} \|u\|_{\frac{2n}{n-2m},\Omega\cap B_2(x)} = 0$.

We state our main result.

THEOREM 4.1. *Under conditions (1) - (3) on f, problem (II) has a classical decaying solution u with $D^\nu u \to 0$ as $|x| \to \infty$ for $|\nu| \leq 2m-1$. If furthermore, $\Omega = R^n$, then u is positive.*

PROOF: In view of Lemma 4.1 we can apply the Mountain Pass Theorem to obtain a nontrivial critical point of J, say u. We omit the proof since it is the same as that of Theorem 3.1. Lemma 4.2 guarantees that u is a classical decaying solution of (II). Suppose $\Omega = R^n$, to see the positivity of u, we observe that $f(x,u) \geq 0$ (here $f(x,t) = 0$ for $t \leq 0$ is assumed), and $\prod\limits_{i=1}^{m-1}(-\Delta + b_i')u \equiv v$ is a solution of the following problem

$$\begin{cases} (-\Delta + b_m')v = f(x,u), & x \in R^n, \\ \lim_{|x|\to\infty} v = 0. \end{cases}$$

If we choose x_0 and apply the Maximum Principle on $B_r(0)$, r large, we conclude: $v(x_0) \geq \inf\limits_{|x|=r} v_-(x)$. Letting $r \to \infty$, we obtain: $\prod\limits_{i=1}^{m-1}(-\Delta + b_i')u = v \geq 0$. Similarly, we can show that $\prod\limits_{i=1}^{m-2}(-\Delta + b_i')u \geq 0, \ldots, (-\Delta + b_1')u \geq 0$, and then $u > 0$, since u is nontrivial.

Unfortunately, for the case of $\Omega \neq R^n$, we are not able to show that the solution u is positive, as we don't have enough information about u on the boundary $\partial\Omega$ (we know only $Du|_{\partial\Omega} = 0$ for $|\nu| \leq m-1$).

Under a further condition on f, we can obtain infinitely many solutions.

THEOREM 4.2. *Assume conditions (1) - (3) with* $\Omega_0 = \Omega$ *in (1). If* f *is odd in* t, *then problem (II) has infinitely many nontrivial decaying solutions whose derivatives up to* $2m-1$ *decay at* ∞.

The proof is essentially the same as the one of Theorem 3.2. We omit it.

4.3. Other problems.

In this section, we study the sublinear problem:

$$\begin{cases} \ell u = h(x)u^{\beta}, & x \in \Omega \\ D^{\nu}u|_{\partial\Omega} = 0, & |\nu| \leq m-1 \\ \lim\limits_{|x|\to\infty} u = 0, \end{cases} \tag{4.3.1}$$

$0 \leq \beta < 1$ and the mixed sub-superlinear problem:

$$\begin{cases} \ell u = g(x)u^{\alpha} + h(x)u^{\beta}, \\ D^{\nu}u = 0, & |\nu| \leq m-1 \\ \lim\limits_{|x|\to\infty} u = 0, \end{cases} \tag{4.3.2}$$

$0 \leq \beta < 1 < \alpha < \frac{n+2m}{n-2m}$.

Kusano and Swanson [KS] considered (4.3.1) in the case of $m = 2$, $\Omega = R^n$, and $h(x) = h(|x|)$. Later Kusano, Naito and Swanson [KNS2] studied both (4.3.1) and (4.3.2). The existence of a positive decaying solution for (4.3.1) and

(4.3.2) was obtained under the assumptions (4.1.1), (4.1.2) and (4.1.3). Their method involves fixed point theory arguments which enable them to deal with the singular case $-1 < \beta < 0$.

We can also deal with the problems (4.3.1) and (4.3.2) in the nonradial case by using the variational approach in the space E. Specifically, we establish certain integration conditions for (4.3.1) and (4.3.2) to have a nontrivial solution. However, the case $-1 < \beta < 0$ does not seem accessible to our method.

For the sublinear problem, we impose the following conditions

(4) $f \in C^{\alpha}_{\text{loc}}(\Omega \times R^+)$, $0 \leq f(x,t)$ for $(x,t) \in \Omega \times R^+$, and $f(x,t) \leq h(x)|t|^{\beta}$, $0 \leq \beta < 1$, $h \in L^{\infty}(\Omega) \cap L^{q_0}(\Omega)$, $q_0 = \frac{2n}{2n-(\beta+1)(n-2m)}$;

(5) $f(x,t) \geq h_0(x)t^{\beta_0}$ as $t \to 0^+$, where $h_0(x) \geq 0$, $\not\equiv 0$, $0 \leq \beta_0 \leq \beta$.

THEOREM 4.3. *Under conditions (4) and (5) Problem (II) has a decaying solution u with $D^{\nu}u \to 0$ as $|x| \to \infty$, for $|\nu| \leq 2m-1$. If $\Omega = R^n$, the solution u is positive.*

PROOF: We can employ the arguments of Theorem 3.3, to establish the existence of a nontrivial weak solution u. Also the regularity and decay of u follows from the arguments of Lemma 4.2 with the replacements of g by h and α by 1 for $\beta > 0$. The case $\beta = 0$ is even simpler. We just apply the Hölder's Inequality to $\int_{\Omega \cap B_2(x)} h|\varphi|$ and choose suitable p_1 and p_1' where $\frac{1}{p_1} + \frac{1}{p_1'} = 1$.

We have the following existence criterion for mixed sub-superlinear problem

THEOREM 4.4. *Let $f(x,t) = f_1(x,t) + f_2(x,t)$, where f_1 satisfies (1) - (3) and f_2 satisfies (4). Then problem (II) has a decaying solution u provided*

$$2A_m^2 \|g\|_{p_0}^{\frac{1-\beta}{\alpha-\beta}} \|h\|_{q_0}^{\frac{\alpha-1}{\alpha-\beta}} \left[\frac{1}{\alpha+1}\left[\frac{(\alpha+1)(1-\beta)}{(\beta+1)(\alpha-1)}\right]^{\frac{\alpha-1}{\alpha-\beta}} + \frac{1}{\beta+1}\left[\frac{(\beta+1)(\alpha-1)}{(\alpha+1)(1-\beta)}\right]^{\frac{1-\beta}{\alpha-\beta}}\right]$$
$$< 1$$

where $A_m = n^{-(m+1)/2}\left(\frac{n-1}{\sqrt[n]{v_n}}\right)^m \frac{\Gamma(n/2-m)}{\Gamma(n/2)}$, the Sobolev embedding constant (see [Ma]), v_n = volume of $B_1(0)$. If $\Omega = R^n$, then the solution u is positive.

PROOF: The same arguments as those of Theorem 3.4 give the existence of a nontrivial critical point u. To show the regularity and decay of u, we only need a simple modification. Note that in the iteration of the proof of Lemma 4.2, one more term in involved in this case:

$$\left|\int_{\Omega\cap B_2(x)} f_2(\cdot,u)\varphi\right| \le \int_{\Omega\cap B_2(x)} h|u|^\beta|\varphi|$$
$$\le \begin{cases} C\|u^\beta\|_{L^{p_1}(\Omega\cap B_2(x))}\|\varphi\|_{p_1'}, & \text{if } \beta > 0 \\ \|h\|_{L^{p_1}(\Omega\cap B_2(x))}\|\varphi\|_{p_1'}, & \text{if } \beta = 0 \end{cases}$$

where $p_1 = \frac{q_1}{\alpha}$, $q_1 = \frac{2n}{n-2m}$ the same as before. But

$$\|u^\beta\|_{L^{p_1}(\Omega\cap B_2(x))} \le C\|u\|^\beta_{L^{q_1}(\Omega\cap B_2(x))}$$

as $\frac{\beta}{\alpha}q_1 < q_1$.

4.4. Remarks and Examples.

We now give some examples showing the connection with earlier work. Some of the examples are in radially symmetric cases, for illustrative purposes, even though our criteria do not require such assumption. For simplicity, we assume all coefficients are smooth.

Example 1. Consider the superlinear biharmonic problem

$$\begin{cases} \Delta^2 u = g(|x|)u^\alpha, & x \in R^n \\ \lim_{|x|\to\infty} u = 0 \end{cases} \tag{4.4.1}$$

where $n \geq 5,\ 1 < \alpha < \frac{n+4}{n-4},\ g \in L^\infty,\ 0 < g(|x|) = 0(|x|^{-a})$ at ∞.

Conditions (1) and (3) are satisfied. If $\frac{2na}{2n-(\alpha+1)(n-2)} > n$, i.e. $a > \frac{n-4}{2}\left[\frac{n+4}{n-4} - \alpha\right]$, then condition (2) is also satisfied. We conclude that (4.4.1) has a positive solution by Theorem 4.1.

As mentioned before, to the best of our knowledge only Dalmasso [Dal] studied the superlinear biharmonic problem. The existence of a positive solution to (4.4.1) was obtained there by the assumption:

$$\int_0^\infty g(r)r^3 dr < \infty.$$

If $g(|x|) = 0(|x|^{-a})$, this condition requires $a > 4$, which is much stronger than ours: $a > \frac{n-4}{2}\left[\frac{n+4}{n-4} - \alpha\right]$. Note that $4 > \frac{n-4}{2}\left[\frac{n+4}{n-4} - \alpha\right] > 0$ for $1 < \alpha < \frac{n+4}{n-4}$.

Example 2. Consider the general superlinear polyharmonic problem

$$\begin{cases} (-\Delta)^m u = g(x)u^\alpha, & x \in R^n \\ \\ \lim\limits_{|x|\to\infty} D^\nu u = 0, & 0 \leq |\nu| \leq 2m-1 \end{cases} \tag{4.4.2}$$

where $n > 2m,\ 1 < \alpha < \frac{n+2m}{n-2m},\ 0 < g(x) \in L^\infty(R^n) \cap L^{p_0}(R^n),\ p_0 = \frac{2n}{2n-(\alpha+1)(n-2m)}$.

In this case, all the conditions of Theorem 4.1 are satisfied, thus (4.4.2) has a classical positive solution. We note in passing that if we replace u^α by $|u|^{\alpha-1}u$

in (4.4.2), then by Theorem 4.2, the corresponding problem has infinitely many decaying solutions, of which at least one is positive. Observe that this example answers the open question posed by Kusano, Naito and Swanson [KNS2].

Example 3. Consider the sublinear problem

$$\begin{cases} (-\Delta)^m u = h(x)u^\beta, & x \in R^n \\ \lim\limits_{|x|\to\infty} D^\nu u \quad \text{[illegible]} & \nu| \le 2m-1 \end{cases} \tag{4.4.3}$$

where $n > 2m$, $0 \le \beta < 1$. Assume h [illegible], and $h \in L^\infty(R^n) \cap L^{q_0}(R^n)$, $q_0 = \frac{2n}{2n-(\beta+1)(n-2m)}$.

By Theorem 4.3, problem (4.4.3) has a positive solution. In [KNS2], Kusano, Naito and Swanson also considered this type of problem with radial case and obtained the existence of a positive solution by assuming $h(x) = h(|x|)$

$$\int_0^\infty h(r) r^{N-1-\beta(N-2m)} dv < \infty,$$

(see Theorem 4, Section 4 of [KNS2]). For $h(|x|) \sim |x|^{-a}$ at ∞, our $h \in L^{q_0}$ requires

$$a > n - \frac{\beta+1}{2}(n-2m),$$

while theirs requires

$$a > n - \beta(n-2m).$$

Whence $h \in L^{q_0}$ is a weaker condition even in this (radial) case.

Example 4. Consider

$$\begin{cases} (-\Delta)^m u = \lambda(g(x)u^\alpha + h(x)u^\beta), & x \in R^n \\ \lim\limits_{|x|\to\infty} u = 0 \end{cases} \tag{4.4.4}$$

with $n > 2m$, $0 \leq \beta < 1 < \alpha < \frac{n+2m}{n-2m}$. Assume that $g \geq 0$, $\not\equiv 0$, and $g \in L^\infty \cap L^{p_0}(R^n)$, $p_0 = \frac{2n}{2n-(\alpha+1)(n-2m)}$, $0 \leq h \in L^\infty \cap L^{q_0}(R^n)$, $q_0 = \frac{2n}{2n-(\beta+1)(n-2m)}$. Then there exists a $\lambda_0 > 0$, for $0 < \lambda \leq \lambda_0$

$$2\lambda_0 A_m^2 \|g\|_{p_0}^{\frac{1-\beta}{\alpha-\beta}} \|h\|_{q_0}^{\frac{\alpha-1}{\alpha-\beta}} \left[\frac{1}{\alpha+1}\left[\frac{(\alpha+1)(1-\beta)}{(\beta+1)(\alpha-1)}\right]^{\frac{\alpha-1}{\alpha-\beta}} + \frac{1}{\beta+1}\left[\frac{(\beta+1)(\alpha-1)}{(\alpha+1)(1-\beta)}\right]^{\frac{1-\beta}{\alpha-\beta}}\right]$$

$$< 1.$$

By Theorem 4.4, for all $0 < \lambda \leq \lambda_0$, problem (4.4.4) has a positive solution u_λ. Note that the conclusion is true for all $\lambda > 0$ if $h \equiv 0$, and we recover the pure superlinear result.

We mention that Theorem 4.4 does not seem easily comparable to Theorem 5 of Section 4 in [KNS2].

CHAPTER 5

SECOND ORDER ELLIPTIC EIGENVALUE PROBLEMS

5.1. Introduction.

We are now concerned with the second order semilinear elliptic eigenvalue problems

$$\begin{cases} \ell u = \lambda f(x,u), & x \in \Omega \\ u|_{\partial\Omega} = 0, & \lim\limits_{|x|\to\infty} u = 0 \end{cases} \tag{III}$$

in a possibly unbounded smooth domain $\Omega \subseteq R^n$, $n \geq 3$, where ℓ denotes the uniformly elliptic operator formally defined by

$$\ell u = -\sum D_i(a_{ij}(x)D_j u) + b(x)u,$$

and $\lambda > 0$.

The object of this chapter is to establish the existence of multiple positive decaying solutions to (III). Basically, we require that the function $f(x,t)$ have multiple positive "humps" in the variable t. The work is motivated by recent results of Noussair and Swanson [NS4] (see also [NS5]), and by earlier results of Brown and Budin [BB], later extended by Hess [He] for bounded Ω. Brown and Budin [BB] considered (III) for bounded Ω when f has multiple humps:

(a) $f(x,0) > 0$;

(b) $f(x,a_i) \leq 0$ for a_i, $1 \leq i \leq N : 0 \equiv a_0 < a_1 < \cdots < a_N$;

(c) $F(x,s_i) > F(x,s)$ for some $s_i : a_{i-1} < s_i \leq a_i$ and $(x,s) \in \overline{\Omega} \times [0,a_{i-1}]$, where $F(x,s)$ is the primitive of $f(x,t)$.

Roughly speaking, the conditions (b) and (c) implies that $f(\cdot,t)$ as a function of t has N positive humps and $N-1$ negative humps. The area of each positive hump is larger than that of previous negative one. Brown and Budin showed the existence of N positive solutions by the variational method. Under the further condition that f is differentiable in t, Hess [He] improved this result and proved the existence of other $N-1$ positive solutions via a degree result of Rabinowitz [Ra2]. In [NS4], Noussair and Swanson considered (III) for Ω unbounded and f with a single hump, i.e. $N=1$ in (b) and (c). They established the existence of two solutions by means of arguments based on the Mountain Pass Theorem. Their procedure involved, in particular, the assumptions: $b(x) \geq b_0 > 0$ and $f(x,0)=0$, f is superlinear near $t=0$. Note that as mentioned earlier the condition $b(x) \geq b_0 > 0$ is significant in the case Ω unbounded, since the operator ℓ induce a equivalent norm in Sobolev space $W_0^{1,2}(\Omega)$, while this is not the case if $b(x) \equiv 0$.

We employ arguments based on the Mountain Pass Theorem to extend the results of Brown and Budin [BB], and Hess [He] to the case Ω unbounded. We first consider the case of Ω bounded and then proceed to consider Ω unbounded. In both cases, the existence of multiple positive decaying solutions is obtained without assuming $b(x) \geq b_0 > 0$, f superlinear or $f \in C^1$. The superlinearity of f at $t=0$ is one of the situation mentioned in [NS4] as being open. We point out, however, that the localization results of Hess [He]:

$$a_{i-1} < \|u_i\|_\infty,\ \|\hat{u}_i\|_\infty < a_i \qquad i=1,\ldots,N$$

where u_i and $\hat{u}_i$ are a pair of solutions, and the decay law of Noussair and Swanson [NS4]:

$$|u(x)| \le Ce^{-\delta|x|}$$

do not immediately follow from our work. Finally, we mention that the degree method of Hess [He] could also be used.

5.2. Assumptions.

Let $F(x,t) = \int_0^t f(x,s)ds$. We assume the following conditions:

(1) $f \in \mathrm{Lip}_{\mathrm{loc}}(\overline{\Omega} \times R^+)$:

(2) $f(x,0) \ge 0$, for $x \in \overline{\Omega}$;

(3) There exist constants $a_1, \dots, a_N$ such that $0 \equiv a_0 < a_1 < \cdots < a_N$ and:

(a) $f(x,a_i) \le 0$ for $x \in \overline{\Omega}$, $i = 1, \dots, N$;

(b) for $i = 1, \dots, N$, there exist a bounded smooth subdomain $D \subseteq \Omega$ and constants $\beta_i > 0$ such that:

$$\int_D F(x,a_i) - \sup_{\substack{v \in C_0^1(\Omega) \\ 0 \le v \le a_{i-1}}} \left[\int_\Omega F(x,v) \right] \ge \beta_i \tag{5.2.1}$$

and $F(x,a_i) - F(x,t) \ge 0$ for $x \in D$, $0 \le t \le a_i$.

It will be obvious from the given proofs that not all of the above conditions are required in every result, however they are globally assumed to avoid fragmenting the presentation. We also observe that the constants a_i could be replaced by suitable supersolutions, while (3) implies $f(x, a_i) = 0$ for $x \in D$, by the increasing property of $F(x,t)$ at $t = a_i$.

We first observe the following elementary implications.

LEMMA 5.1.

(a) For any bounded subset $K \subset \overline{\Omega} \times R^+$ *there exists a constant* $k = k(K)$ *such that* $f(x,t) + kt$ *is nondecreasing in* t.

(b) Let:

$$\alpha_i = \inf_{\overline{D}\times[0,a_{i-1}]} (F(x,a_i) - F(x,t))$$

$$\gamma_i = \sup_{\substack{v \in C_0^1(\Omega) \\ 0 \leq v \leq a_{i-1}}} \Big[\int_{\Omega \setminus D} F(x,v) \Big].$$

If $\alpha_i \mathrm{mes}(D) > \gamma_i$ *then inequality (5.2.1) holds for some* β_i.

(c) If $N = 1$ *and* $f(x_0, a_1) > 0$ *for some* x_0 *then inequality (5.2.1) holds.*

(d) Assume that Ω *is bounded,* $f(x,t) = f(t)$ *and there exist constants* $0 \equiv a'_0 < a'_1 < \cdots < a'_N$ *with* $f(a'_i) \leq 0$ *for* $i = 1, \ldots, N$, *and* $\max\{F(t),\ 0 \leq t \leq a'_{i-1}\} < F(a'_i)$, *then (3) holds.*

PROOF: Note that (a), (b) and (c) are immediate while (d) follows from simple arguments if we set $G(t) = F(t) - F(a'_i)$ for $t \in [a'_i, a'_{i+1}]$ and choose $a_{i+1} = \inf\{t \mid G$ assumes its absolute maximum on $[a'_i, a'_{i+1}]$ at $t\}$.

5.3. Bounded Domain Results.

In this section we assume Ω bounded and consider problem (III) on Ω. The decay condition is void in this case. Note that these results are valid -with obvious changes – even if $n = 2$. Our first result is the existence of N solutions and is similar to one in [BB] by Brown and Budin. We sketch the proof for completeness and convenience.

THEOREM 5.1 *Under conditions (1) – (3), there exists a λ^* such that for $\lambda \geq \lambda^*$, problem (III) has at least N positive solutions $\{u_i(\lambda, x)\}_{i=1}^{N}$ with $a_{i-1} < \|u_i\|_\infty < a_i$ for $i = 1, \ldots, N$.*

PROOF: Let $f_i(x,t)$ denote the truncation of f:

$$f_i(x,t) = \begin{cases} f(x,0) & t < 0 \\ f(x,t) & 0 \leq t \leq a_i \\ f(x,a_i) & t > a_i, \end{cases}$$

and consider $\ell u = \lambda f_i(x,u)$ in $W_0^{1,2}(\Omega)$. By the assumptions on f, the functional

$$J_i(\lambda, u) = \frac{1}{2}\|u\|_\ell^2 - \lambda \int_\Omega F_i(x,u)dx,$$

where $\|u\|_\ell^2 = \int_\Omega \sum a_{ij} D_i u D_j u + bu^2$, $F_i(x,t) = \int_0^t f_i(x,s)ds$ is coercive, C^1 and weakly lower semicontinuous on $W_0^{1,2}(\Omega)$, whence it attains its minimum at some point of $W_0^{1,2}(\Omega)$, say u_i. The maximum principle applied to $\ell + k$ shows that $0 < u_i < a_i$ in Ω, and it follows that u_i is a solution of (III). Finally, for λ sufficiently large, there exists $x_0 \in \Omega$ such that $u_i(\lambda, x_0) > a_{i-1}$ for if this is not the case, let: $g(x) = a_i$ if $B_{2h}(x) \subset D$, $g(x) = 0$ otherwise. Consider $g_h(x) =$ mollifer of g and note that $g_h \in W_0^{1,2}(D)$ and, since $F_i(x,0) = 0$, we have

$$\int_D F(x, g_h) = \int_\Omega F_i(x, g_h).$$

We conclude:

$$
\begin{aligned}
&J_i(\lambda, g_h) - J_i(\lambda, u_i)\\
&\le \frac{1}{2}\|g_h\|_\ell^2 - \lambda\Big(\int_D F_i(x, g_h) - \int_\Omega F_i(x, u_i)\Big)\\
&\le \frac{1}{2}\|g_h\|_\ell^2 - \lambda(\beta_i - 0(h))
\end{aligned}
$$

by (5.2.1). Choose first h small and then λ large shows that $J_i(\lambda, g_h) < J_i(\lambda, u_i)$ contradicting the minimality of $J_i(\lambda, u_i)$.

We now show the existence of more solutions by means of the Mountain Pass Theorem. We assume that the solution $\{u_i\}$ found in Theorem 5.1 are isolated, for otherwise there is nothing to show. We know from the proof of Theorem 5.1 that u_i is the absolute minimum of J_i. Naturally we conjecture that u_i must be a local minimum of J_{i+1}. If so, as u_{i+1} is the absolute minimum of J_{i+1} and $J_{i+1}(u_{i+1}) < J_{i+1}(u_i) = J_i(u_i)$, by the Mountain Pass Theorem, we can claim the existence of another solution "between" u_i and u_{i+1}, say u^*_{i+1}, with $J_i(u_{i+1}) < J_{i+1}(u^*_{i+1})$. An estimate below will show that $J_{i+1}(u^*_{i+1}) < J_{i-1}(u_{i-1})$. Then the existence of other $N-1$ distinct solution $\{u^*_i\}_{i=2}^N$ follows. These are the basic ideas. The proof will be accomplished by first establishing several intermediate results.

LEMMA 5.2. *Assume $\lambda > 0$ is sufficiently large and that u_i is the only solution in $B_r(u_i)$ for $i = 1, \dots, N$.*

(a) There exists a function $c : [0, \varepsilon^*] \to R^+$ *for some* $\varepsilon^* > 0$ *such that* $c(t) > 0$ *if* $t > 0$ *and if* $\varepsilon \le \|u - u_i\|_\ell \lesssim \tau/2$ *then*

$$J_i(u) \ge J_i(u_i) + c(\varepsilon); \tag{5.3.1}$$

(b) For $\varepsilon > 0$ *sufficiently small, there exist* $\delta > 0$ *such that if* $\varepsilon = \|u - u_i\|_\ell$ *then*

$$J_{i+1}(u) \ge J_{i+1}(u_i) + \delta; \tag{5.3.2}$$

(c) There exist positive constants α_0, β_0 *(independent of* λ, i*) such that*

$$J_i(u_j) \ge J_{i+1}(u_{i+1}) - \alpha_0 + \lambda\beta_0 \tag{5.3.3}$$

for $j \le i$.

Proof: (a) Consider $H(u) = J_i(u) - J_i(u_i)$ and observe that $H \ge 0$, H is continuous and weakly lower semicontinuous. Set

$$c(\varepsilon) = \inf\{H(u) \mid \varepsilon \le \|u - u_i\|_\ell \le \tau/2\}$$

and observe that if $c(\varepsilon) = 0$ for some $\varepsilon > 0$ then there exists a minimizing sequence $\{u_k\}$ for H in $\varepsilon \le \|u_k - u_i\|_\ell \le \tau/2$. Without loss of generality, there exists v in $W_0^{1,2}(\Omega)$ such that $u_k \to v$ weakly in $W_0^{1,2}(\Omega)$ and strongly in L^2. By the properties of f, $F_i(x, u_k) \to F_i(x, v)$ in L^1, whence $\|u_k\|_\ell \to \|v\|_\ell$ by the definition of H. We conclude that v is a solution in $\varepsilon \le \|u - u_i\| \le \tau/2$ and the contradiction establishes the result.

(b) Let $u \in W_0^{1,2}(\Omega)$ and set $\tilde{u} = \min(u, a_i)$. Note that $\tilde{u} \in W_0^{1,2}(\Omega)$ and:

$$\begin{aligned}
J_{i+1}(u) = J_i(\tilde{u}) &+ \frac{1}{2}\int_{\{u>a_i\}} \left[\sum a_{ij}D_iuD_ju + b(u^2 - a_i^2)\right] \\
&- \lambda\int_{\{u>a_i\}} \left[F_{i+1}(x,u) - F_{i+1}(x,a_i)\right] \\
&\geq J_i(\tilde{u}) + \frac{1}{2}\|(u-a_i)^+\|_\ell^2 - \frac{\lambda M}{2}\int_\Omega \chi(u>a_i)[(u-a_i)^+]^2,
\end{aligned}$$

where M denotes the Lipschitz constant for f_i and $\chi(u > a_i)$ is the characteristic function of the set $\{x \mid u > a_i\}$. Consequently,

$$\begin{aligned}
J_{i+1}(u) \geq J_i(\tilde{u}) &+ \frac{1}{2}\|(u-a_i)^+\|_\ell^2 \\
&- \frac{\lambda}{2}M\|\chi(u>a_i)\|_{\frac{n}{2}}\|(u-a_i)^+\|_{\frac{2n}{n-2}}.
\end{aligned}$$

By Sobolev's Theorem, for some C,

$$\begin{aligned}
J_{i+1}(u) \geq J_i(\tilde{u}) &+ \frac{1}{2}\|(u-a_i)^+\|_\ell^2 \\
&- \lambda C\left[\operatorname{mes}(u>a_i)\right]^{\frac{2}{n}}\|(u-a_i)^+\|_\ell^2.
\end{aligned}$$

Observe that since $u_i < a_i$ in Ω, u_i is a classical solution, and

$$\begin{aligned}
\|u-u_i\|_\ell &\geq C\|u-u_i\|_{\frac{2n}{n-2}} \\
&\geq C(a_i - \|u_i\|_\infty)[\operatorname{mes}(u>a_i)]^{\frac{n-2}{2n}},
\end{aligned}$$

choosing $\varepsilon = \|u-u_i\|_\ell$ sufficiently small yields:

$$1 - 2\lambda C[\operatorname{mes}(u>a_i)]^{\frac{2}{n}} > 2C_1 > 0.$$

Finally, observe that:

$$\varepsilon = \|u - u_i\|_\ell \leq \|\tilde{u} - u_i\|_\ell + \|(u - a_i)^+\|_\ell,$$

whence either $\|\tilde{u} - u_i\|_\ell \geq \frac{\varepsilon}{2}$ or $\|(u - a_i)^+\|_\ell \geq \frac{\varepsilon}{2}$, and for ε small enough, $\|\tilde{u} - u_i\|_\ell \leq \frac{r}{2}$. By Part (a) we conclude:

$$J_{i+1}(u) \geq J_1(u_i) + \max\left\{C_1\|(u - a_i)^+\|_\ell^2,\ C(\|u - u_i\|_\ell)\right\},$$

and (5.3.2) follows.

(c) It clearly suffices to estimate $J_i(u_i) - J_{i+1}(u_{i+1})$, but this is an immediate consequence of the arguments used in Theorem 5.1.

LEMMA 5.3. *Let $\varphi \in C_0^\infty(\Omega')$ with $\Omega' \subset C\Omega$, $0 \leq \varphi \leq 1$. Then there exists a constant $K_1 = K_1(\varphi)$, independent of λ and u_i, such that:*

$$\frac{\|\varphi u_i\|_\ell^2}{2} + \lambda \int_\Omega \left[F_i(x, u_i + \varphi(a_i - u_i)) - F_i(x, u_i)\right] \leq K_1(\varphi). \tag{5.3.4}$$

PROOF: Set $G = \int_\Omega \left[F_i(x, u_i + \varphi(a_i - u_i)) - F_i(x, u_i)\right]$ and observe that, by definition,

$$J_i(u_i + \varphi(u_i - u_i)) \geq J_i(u_i).$$

Setting $u_i = u_i(1 - \varphi) + \varphi u_i$ on the right hand side and expanding yield in $W_0^{1,2}(\Omega)$:

$$\langle (1 - \varphi)u_i, \varphi(a_i - u_i)\rangle_\ell + \frac{\|\varphi a_i\|_\ell^2}{2} \geq \frac{\|\varphi u_i\|_\ell^2}{2} + \lambda G.$$

Moreover, again expanding, gives

$$\begin{aligned}
&\langle (1-\varphi)u_i,\ \varphi(a_i-u_i)\rangle_\ell \\
&= \int_\Omega \Big[\sum a_{kj}\big\{(1-\varphi)(a_i-u_i)D_k u_i D_j\varphi - \varphi(1-\varphi)D_k u_i D_j u_i \\
&\qquad - u_i(a_i-u_i)D_k\varphi D_j\varphi + u_i\varphi D_k\varphi D_j u_i\big\} \\
&\qquad + b\varphi(1-\varphi)u_i(a_i-u_i)\Big].
\end{aligned}$$

Since $0 \le \varphi \le 1$ and $0 < u_i < a_i$, we disregard the negative terms and, using the Divergence Theorem, obtain:

$$\begin{aligned}
&\langle (1-\varphi)u_i,\ \varphi(a_i-u_i)\rangle_\ell \\
&\le \int_\Omega \Big[\frac{1}{4}\ell(\varphi^2)u_i^2 + \frac{1}{2}\sum D_k(a_{kj}D_j(\varphi)(1-\varphi))\cdot(a_i-u_i)^2 \\
&\qquad + b\varphi(1-\varphi)u_i(a_i-u_i)\Big]
\end{aligned}$$

and the proof is complete.

LEMMA 5.4. *Set $v(t) = u_i + t\varphi(a_{i+1}-u_i)$ with $0 \le t \le 1$, and φ as in Lemma 5.3. Then for some constant $K = K(\varphi)$,*

$$\begin{aligned}
J_{i+1}(v(t)) \le J_i(u_i) + K(\varphi) - \lambda\int_\Omega &\{F_{i+1}(x,v(t)) \\
&+ F_{i+1}(x,u_i+\varphi(a_i-u_i)) - 2F_{i+1}(x,u_i)\}dx.
\end{aligned} \tag{5.3.5}$$

PROOF: Set $\theta = t\varphi$, and let G represent the integral on the right hand side of (5.3.5). If we expand $\|v(t)\|_\ell^2$ and use the Divergence Theorem as in Lemma 5.3, we conclude:

$$\frac{\|v(t)\|_\ell^2}{2} \le \frac{1}{2}\|u_i\|_\ell^2 + \frac{1}{2}\|\theta u_i\|_\ell^2 + K_2(\varphi).$$

If we now employ Lemma 5.3 we conclude:

$$J_{i+1}(v(t)) \le J_{i+1}(u_i) + [K_1(\varphi) + K_2(\varphi)] - \lambda G.$$

Since $J_{i+1}(u_i) = J_i(u_i)$, inequality (5.3.5) is established.

THEOREM 5.2. *Under conditions (1) - (3). There exists $\lambda^* > 0$, for $\lambda \ge \lambda^*$, problem (III) has at least $2N - 1$ positive solutions.*

PROOF: By Lemma 5.2 (b), u_i is a local minimum of J_{i+1}. Let $e = u_i + \varphi(a_{i+1} - u_i)$, where φ is free for a moment. Choose a path $v(t)$ between u_i and e: $v(t) = u_i + t\varphi(a_{i+1} - u_i)$ with $0 \le t \le 1$. The proof will be immediate from the Mountain Pass Theorem once we show

(a) $J_{i+1}(e) < J_{i+1}(u_i)$ $(= J_i(u_i))$;

(b) $J_{i+1}(v(t)) < J_i(u_i) - \alpha_0 + \lambda\beta_0$, for $0 \le t \le 1$ with α_0, β_0 as in Lemma 5.2 (c). Note that the (P.S.) condition is obvious in this case, and that Part (b) here is only used to show that the new solution is not one previously found. The proof depends on a suitable selection of the function φ.

By Lemma 5.4, we need only estimate the integral on the right hand side of (5.3.5):

$$\begin{aligned}\int_\Omega \{F_{i+1}(x, u_i + t\varphi(a_{i+1} - u_i)) - F_{i+1}(x, u_i)\\ + F_{i+1}(x, u_i + \varphi(a_i - u_i)) - F_{i+1}(x, u_i)\}dx.\end{aligned} \tag{5.3.6}$$

Note that Ω in (5.3.6) can be replaced by $\operatorname{supp}(\varphi)$. We first choose $\operatorname{supp}(\varphi) \subset D$ (see (3) (b)). Let $t = 1$ in (5.3.6). Observe that

$$F_{i+1}(x, a_{i+1}) - F_{i+1}(x, u_i) > 0, \quad x \in \{\varphi = 1\}$$

and

$$|F_{i+1}(x, u_i + \varphi(a_{i+1} - u_i)) - F_{i+1}(x, u_i)| < m, \qquad x \in \operatorname{supp}(\varphi)$$

for some constant M by (3) and (1). We thus select $\operatorname{supp}(\varphi)$ such that for some $c_0 > 0$

$$\int_{\{\varphi=1\}} \left[F_{i+1}(x, a_{i+1}) - F_{i+1}(x, u_i)\right] \geq c_0,$$

while

$$\int_{\{\varphi=1\}} \left[F_{i+1}(x, a_i) - F_{i+1}(x, u_i)\right] \geq 0,$$

and

$$a_N M \ \operatorname{mes}(0 < \varphi < 1) < \frac{c_0}{4}.$$

We then have, by Lemma 5.4,

$$\begin{aligned} J_{i+1}(e) &\leq J_i(u_i) + K(\varphi) - \lambda\Big(c_0 - \frac{c_0}{2}\Big) \\ &\leq J_i(u_i) + K(\varphi) - \lambda \ \frac{c_0}{2}. \end{aligned}$$

Note that φ is chosen independently of λ, u_i. We conclude Part (a) for large λ. By exactly the same argument; we have

$$J_{i+1}(v(t)) \leq J_i(u_i) + K(\varphi) + 2\lambda M a_N \ \operatorname{mes}(\operatorname{supp}(\varphi)).$$

As a final choice, we assume $\operatorname{supp}(\varphi)$ is so small that $2a_N M \operatorname{mes}(\operatorname{supp}(\varphi)) < \beta_0$, we obtain (b) for large λ. Therefore the Mountain Pass Theorem gives the existence of a solution u_{i+1}^* with

$$J_i(u_i) < J_{i+1}(u_{i+1}^*) < J_{i-1}(u_{i-1})$$

for λ large enough.

5.4. Existence of N Decaying Solutions.

We henceforth assume that Ω is unbounded. The existence results of the preceding section easily yield the existence of N solutions even for Ω unbounded. We need only observe that super-subsolution arguments are valid and $\psi = a_i$ gives a supersolution while replacing Ω by $\Omega_r = \Omega \cap \{|x| < r\}$, r large, in Theorem 5.1 and setting $\varphi = u_i$ in Ω_r, $\varphi = 0$ in $\Omega - \Omega_r$, gives a subsolution. For the purposes of showing the existence of decaying solutions, however, and for the applicability of the variational methods used in Section 5.5 it is better to use an approximation argument as the following shows. We denote by $\hat{f}_i(x)$ the function $\sup\limits_{0 \leq t \leq a_i} |f(x,t)|$.

THEOREM 5.3. *Suppose f satisfies (1) - (3), $\sum D_i(a_{ij})$ is bounded, $b(x) \geq \eta(|x|) > 0$ at ∞. Then there exists a number γ such that if $\xi(r)$ is a positive smooth decaying function and*

$$\lim_{r\to\infty} \left\{ \frac{|\xi''(r)| + |\xi'(r)|}{\xi(r)\eta(r)} \right\} \leq \gamma,$$

$$\lim_{|x|\to\infty} \left\{ \frac{\hat{f}_i(x)}{\xi(|x|)\eta(|x|)} \right\} = 0,$$

then problem (III) has for λ sufficiently large, N nonnegative solutions $\{u_i(\lambda,x)\}_{i=1}^{N}$ such that $u_i(\lambda,x) \leq c\xi(|x|)$ for some $c = c(\lambda) > 0$.

Proof: Let $\Omega_k = \{x \in \Omega \mid |x| < k\}$ with k chosen sufficiently large and consider equation (1) on $W_0^{1,2}(\Omega_k)$. For $i \in \{1,\ldots,N\}$, we apply Theorem 5.1 to show the existence of $\{u_i^k(\lambda,x)\}_{i=1}^{N}$, solutions of (III) in $W_0^{1,2}(\Omega_k)$ for λ sufficiently large (independent of k). For notational convenience, we sometimes denote $u_i^k(\lambda,x)$ as u_i^k in the rest of the proof. Recall the estimate $a_{i-1} < \|u_i^k\|_{L^\infty} < a_i$. This bound and the interior and boundary Schauder Estimates imply that for each i and each λ (sufficiently large) there is a subsequence (also denoted by u_i^k) which converges locally uniformly to a solution $u_i(\lambda,x)$ of (III). Finally, let $\omega(x) = c\xi(|x|)$ with c to be chosen later. A direct calculation in $\Omega_k - \Omega_{k_0}$ shows:

$$L(\omega - u_i^k) \geq c\xi(|x|)\eta(|x|)\Big\{1 - \frac{\xi'(|x|)}{\xi(|x|)\eta(|x|)}\Big(\sum\ \frac{a_{ii}}{|x|} + \sum D_i(a_{ij})\frac{x_j}{|x|}\Big)$$

$$- \frac{(\xi''(|x|) - \xi'(|x|)/|x|)}{\xi(|x|)\eta(|x|)}\sum a_{ij}\ \frac{x_i x_j}{|x|^2} - \frac{\lambda}{c}\ \frac{\hat{f}_i(x)}{\xi(|x|)\eta(|x)}\Big\}.$$

Selecting γ sufficiently small, k_0 sufficiently large and $c = \max\left\{\frac{a_N}{\xi(k_0)}, 1\right\}$ we conclude from our coefficient assumptions that $L(\omega - u_i^k) \geq 0$ in $\Omega_k - \Omega_{k_0}$, while $\omega - u_i^k \geq 0$ on $\partial(\Omega_k - \Omega_{k_0})$ by choice of c. We obtain $\omega(x) \geq u_i^k(\lambda,x)$ whence $\omega(x) \geq u_i(\lambda,x)$. Finally, let $u_i^k(\lambda,x_i^k) > a_{i-1}$, and observe that $\{x_i^k\}_{k=1}^{\infty}$ is bounded by the uniform decay of $\{u_i^k\}$. Without loss of generality, we conclude that $x_i^k \to x_i$ for some x_i and, by equicontinuity, $u_i(\lambda,x_i) \geq a_{i-1}$. Suppose now $u_i = u_{i-1}$ for some i. We then must have $u_{i-1}(\lambda,x_i) = a_{i-1}$, and $u_{i-1} \leq$

a_{i-1}. We again apply (3) (b) and the maximum principle to $\ell + k$ to obtain a contradiction. This argument shows that $u_i \neq u_{i-1}$, i.e. that $\{u_i\}_1^N$ are distinct solutions. We need only show $u_1 \not\equiv 0$. Suppose $u_1 \equiv 0$. Let φ be as in the proof of Theorem 5.2. Using the arguments there and observing that $u_1^k \xrightarrow{k} 0$ locally uniformly in Ω, we then conclude:

$$J_1\big(u_1^k + \varphi(a_1 - u_1^k)\big) < J_1(u_1^k)$$

for λ large, contradicting the minimality of $J_1(u_1^k)$. This completes the proof.

We observe the following. If $b(x) \geq b_0 > 0$, we may select $\eta(r) = b_0$, and $\xi(r) = e^{-\beta r}$ or $r^{-\beta}$ or $(\ell n\ r)^{-\beta}$ for some $\beta > 0$. However, if $b(x) \equiv 0$, the conclusion of Theorem 5.3 holds if we assume $\lim_{|x| \to \infty} \left(\sum D_i(a_{ij})x_j\right)_- = 0$ and $\lim_{|x| \to \infty} (\hat{f}_i(x)|x|^{2+\beta}) = 0$ with $0 < \beta < \frac{\nu n}{\delta} - 2$. In such a case, the key estimate in the proof of Theorem 5.3 becomes:

$$\begin{aligned} L(\omega - u_i^k) \geq c|x|^{-(2+\beta)} \Big\{ & \delta\beta\Big(\frac{\nu n}{\delta} - 2 - \beta\Big) \\ & + \beta\Big(\sum D_i(a_{ij})x_j\Big)_- - \frac{\lambda}{c}\hat{f}_i(x)|x|^{2+\beta} \Big\} \end{aligned}$$

and we conclude (in the same way) that $u_i \leq c|x|^{-\beta}$.

5.5. Existence of $2N - 1$ Decaying Solutions.

In this section we assume $\hat{f}_i$ belongs to a suitable weighted space. This new condition enables us to give a variational characterization to the N solutions found earlier and, furthermore, to show the existence of $N - 1$ other solutions. We are also able to treat the case $b \geq 0$ without any assumptions on the behaviour of b or (a_{ij}) at ∞. We can not, however, give as precise a decay formula for the solutions we now find.

We also remark that it may appear that the existence of $2N-1$ solutions follows immediately from Theorem 5.2 by using the same finite domain approximation arguments as we used in Theorem 5.3. This is not the case, however, since from the estimate $a_{i-1} < \|u_i^k\|_\infty < a_i$ we can only conclude $a_{i-1} \leq \|u_i\|_\infty \leq a_i$, hence, we are unable to show that, in the limit as $k \to \infty$, solutions remain distinct.

The main idea involves the creation of a structure analogous to that used in Section 5.3 for the bounded domain case, so that the same theory can be used.

Let the dimension $n \geq 3$ and choose $E = E(1)$ as the function space. Recall that the norm of E is $\|u\|_\ell = \left(\int_\Omega \sum a_{kj} D_k u D_j u + bu^2\right)^{\frac{1}{2}}$, and Sobolev's Inequality $\|u\|_{\frac{2n}{n-2}} \leq C\|u\|_\ell$ holds.

We first prove a property of E.

LEMMA 5.5. *If $u \in E \cap C^1(\Omega)$, then there exists a sequence $\varphi_k \in C_0^1(\Omega)$ such that $\varphi_k \to u$ in E and $\|\varphi_k\|_\infty \leq \|u\|_\infty$.*

PROOF: Let $\varphi_k(x) = u(x) \cdot \theta\left(\frac{|x|}{k}\right)$, where $\theta(t) \in C^1(R^+)$ such that $0 \leq \theta(t) \leq 1$, $\theta(t) = 1$ for $0 \leq t \leq 1$ and $\theta(t) = 0$ for $t \geq 2$. Then $\varphi_k \in C_0^1(\Omega)$ and

$\|\varphi_k\|_\infty \leq \|u\|_\infty$. To see $\varphi_k \to u$ in E, note that

$$\begin{aligned}\|\varphi_k - u\|_\ell^2 &\leq C\int_\Omega |\nabla\Big(u\theta\Big(\frac{|x|}{k}, -u\Big)|^2 + b\Big(u\theta\Big(\frac{|x|}{k}\Big) - u\Big)^2 \\ &\leq C\int_{k\leq|x|\leq 2k} |\Big(\theta\Big(\frac{|x|}{k}\Big) - 1\Big)\nabla u - \frac{1}{k}\theta'\Big(\frac{|x|}{k}\Big)u\nabla|x|\ |^2 + \int_{k\leq|x|} bu^2 \\ &\leq C\int_{k\leq|x|} (|\nabla u|^2 + w_0u^2)\end{aligned}$$

$w_0 = \max\left\{b,\ \frac{1}{1+|x|^2}\right\}$, since $\frac{1}{k^2} \leq \frac{5}{1+|x|^2}$ for $k \leq |x| \leq 2k$. This concludes the proof by the definition of E.

We now introduce the final condition on f.

(4) $f_i(x,t)$ is Lipschitz in t uniformly with respect to x, and $\hat{f}_i \in L^\infty(\Omega)\cap L^{\frac{2n}{n+2}}(\Omega)$.

Note that $L^{\frac{2n}{n+2}}(\Omega)$ can be replaced by weighted space $L^2_{w_0^{-1}}(\Omega)$ as the proof below shows.

As before we define

$$J_i(\lambda,u) = \frac{1}{2}\|u\|_\ell^2 - \lambda\int_\Omega F_i(x,u)dx.$$

for $u \in E$. J_i is well-defined and coercive as

$$\begin{aligned}|\int_\Omega F_i(x,u)| &\leq \int_\Omega \hat{f}_i|u| \leq \|\hat{f}_i\|_{\frac{2n}{n+2}}\|u\|_{\frac{2n}{n-2}} \\ &\leq C\|\hat{f}_i\|_{\frac{2n}{n+2}}\|u\|_\ell.\end{aligned}$$

Furthermore, $J_i(\lambda,\cdot)$ is weakly lower semi-continuous and C^1 on E, $\int_\Omega f_i(x,u)\cdot\varphi\ dx$ – the derivative of $\int_\Omega F_i(x,u)dx$ – is a compact map from E to E. The proofs are basically the same as that of Lemma 3.1 in Chapter 3.

Set $N_i(\lambda) = \{u \mid u \in E \text{ and } J_i'(u) = 0\}$.

Lemma 5.6. *Under conditions (1) – (4),*

(a) There exists $R = R(\lambda)$, independent of i, such that $N_i(\lambda) \subset B_R(0)$;

(b) If $u \in N_i(\lambda)$ then $0 < u < a_i$ and $\lim_{|x|\to\infty} u = 0$;

(c) There is $u_i \in N_i(\lambda) \setminus N_{i-1}(\lambda)$ $(N_0 \equiv \{0\})$ such that

$$J_i(\lambda, u_i) = \inf\{J_i(\lambda, u) \mid u \in E\}.$$

Proof: (a) Note that if $u \in N_i(\lambda)$, then $\|u\|_E^2 = \lambda \int_\Omega f_i(x,u)u \le \lambda C \|\hat{f}_i\|_{\frac{2n}{n+2}} \|u\|_{\frac{2n}{n-2}}$.

(b) Let $u \in N_i(\lambda)$, $0 < u < a_i$ is immediate from (2) and (3) (a). By Theorem 8.25 of [GT]

$$u_i(x) \le C\Big(\|u\|_{L^{\frac{2n}{n-2}}(B_1(x)\cap\Omega)} + \|\hat{f}\|_{L^q(B_1(x)\cap\Omega)}\Big)$$

for some $q > \frac{n}{2}$. The decay of u_i follows.

(c) Observe that the solutions $\{u_i^k\}$ – constructed in the proof of Theorem 5.3 – form a minimizing sequence for J_i by Lemma 5.5, and $\|u_i^k\| \le C$ by Part (a). Since the ball is weakly compact, we conclude that a subsequence of $\{u_i^k\}$ converges weakly to u_i in E and pointwise, while by the continuity and weakly lower semicontinuity of J_i we find $J_i(\lambda, u_i) = \inf\{J_i(\lambda, u) \mid u \in E\}$, which implies $u_i \in N_i$. Since $J_i(\lambda, u_i) < J_{i-1}(\lambda, u_{i-1})$ by (3), whence $u_i(x_0) > a_{i-1}$ for some x_0, it follows that $u_i \in N_i \setminus N_{i-1}$.

THEOREM 5.4. *Under conditions (1) - (4), there exists* λ^**, such that for* $\lambda \geq \lambda^*$*, problem (III) has* $2N-1$ *positive decaying solutions.*

PROOF: We have obtained the existence of N solutions $\{u_i\}_{i=1}^N$ in Lemma 5.6. Observe that the (PS) condition holds from the compactness of $\int_\Omega f_i(x,u)\varphi\, dx$, also observe that the conclusions of Lemma 5.3 and 5.4 are still true here since the proofs are local in nature, we conclude, exactly in the same way as for Ω bounded, the existence of other $N-1$ solutions.

Note:

(a) If furthermore $f(x,0) \equiv 0$ and $f(x,t)$ is superlinear at $t=0$, under an additional integration condition (see the superlinear case of Section 3.2) it is easy to prove that the problem (III) has one more solution "between" 0 and u_1, that is: problem (III) has $2N$ solutions.

(b) We can also consider the problem (III) in E by following the lines of [He] under the same conditions above. Note that the condition $f \in C^1$ imposed by [He] could be replaced by $f \in \mathrm{Lip}_{\mathrm{loc}}(\overline{\Omega} \times R^+)$. See, e.g., [Am].

5.6. Examples.

We conclude by illustrating our results and the above remark with the following simple example:

Example. Consider the problem:

$$\begin{cases} -\Delta u = \lambda g(x) u^r (\sin u)^*, & x \in \Omega \\ u|_{\partial\Omega} = 0, \text{ (and } \lim\limits_{|x|\to\infty} u = 0 \text{ is unbounded)} \end{cases} \tag{5.6.1}$$

(i) $0 < r$;

(ii) $(\sin\xi)^* = \begin{cases} \sin\xi & 0 < \xi < 3\pi \\ 0 & \text{otherwise}; \end{cases}$

(iii) $g \in L^1 \cap L^\infty \cap C^1(\Omega)$; g changes sign, $g \nrightarrow 0$ as $|x| \to \infty$ if Ω is unbounded.

In this case, we set $a_i = (2i-1)\pi$ for $i = 1, 2$, $B = \{x \mid g(x) \geq 0\}$. Observe that $f(x,u) = gu^r(\sin u)^*$ satisfies (1) - (4). To see this, let $v(x)$ be such that $0 \leq v \leq a_1 = \pi$. Then

$$\int_B F_2(x,a_2) - \int_\Omega F_2(x,v) \geq \int_B g_+(x)\Big[\int_0^{3\pi} t^r \sin t\Big] - \int_B g_+(x)\Big[\int_0^{\pi} t^r \sin t\Big]$$

$$= \int_B g_+(x)\Big(\int_0^{\pi} (\sin t)((2\pi + t)^r - (\pi + t)^r)\Big)dtdx > 0.$$

We need only select $D = B \cap \{x \mid |x| < r_0\}$ for r_0 sufficiently large. We can now apply Theorem 5.4 to conclude that (5.6.1) has three positive decaying solutions for λ large. Note that this example does not seem ameanable to the procedures of [NS4] or [NS5]. We conclude by remarking that it is not difficult to ensure there exists yet another solution "between" 0 and u_1. In this example we need only ensure $J_1(u) > \varepsilon > 0$ for $\|u\|_\ell$ small. Suppose, for example, condition (i) is strengthened to: $1 < r < (n+2)/(n-2)$. We have

$$J_1(u) \geq \|u\|_\ell^2 - \lambda \int_\Omega |g(x)| \; \Big|\int_0^u t^r (\sin t)^* dt\Big| dx$$

$$\geq \frac{1}{2}\|u\|_\ell^2 - \frac{\lambda}{r+1}\|g\|_{L^{q_0}(\Omega)} \|u\|^{r+1}_{L^{\frac{2n}{n-2}}(\Omega)}$$

$$\geq \frac{1}{2}\|u\|_\ell^2 - \lambda \; \frac{c}{r+1}\|g\|_{L^{q_0}(\Omega)} \|u\|_\ell^{r+1}$$

where $q_0 = \frac{2n}{2n-(r+1)(n-2)}$ ($n \geq 3$ then $q_0 > 1$, so $g \in L^{q_0}(\Omega)$). Then $J_1(u) \geq \alpha > 0$ for all $u \in \partial B_\rho(0)$ some small $\rho > 0$ since $r + 1 > 2$.

CHAPTER 6

p-LAPLACIAN PROBLEMS

6.1. Introduction.

This Chapter is devoted to the study of the so-called p-Laplacian problems:

$$\begin{cases} \ell u = f(x,u) \quad \text{in} \quad \Omega, \\ \\ u|_{\partial\Omega} = 0, \ \lim\limits_{|x|\to\infty} u = 0 \end{cases} \tag{IV}$$

where Ω is a smooth exterior domain (with compact boundary) in R^n, ℓ is the p-Laplacian operator defined by

$$\ell u = -\mathrm{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big) + b(x)|u|^{p-2}u,$$

$1 < p < n$, $0 < a_0 \leq a(x) \in L^\infty(\Omega) \cap C^1(\Omega)$, $0 \leq b(x) \in L^\infty(\Omega) \cap C^0(\Omega)$. The objective is to obtain sufficient conditions on f for (IV) to have positive solutions in the following three prototype cases:

(a) $f(x,u) = g(x)u^\alpha, \quad p-1 < \alpha < p^* - 1;$ (6.1.1)

(b) $f(x,u) = h(x)u^\beta, \quad 0 \leq \beta < p-1;$ (6.1.2)

(c) $f(x,u) = g(x)u^\alpha + h(x)u^\beta, \quad 0 \leq \beta < p-1 < \alpha < p^* - 1,$ (6.1.3)

where $p^* = \frac{np}{n-p}$, the Sobolev critical exponent. Note that when $p = 2$, (a), (b) and (c) correspond to the superlinear, sublinear and mixed sub-superlinear problems respectively. By analogy, we call the cases (6.1.1), (6.1.2) and (6.1.3) superhomogeneous, subhomogeneous and mixed sub-superhomogeneous respectively.

For $p = 2$, problem (IV) is the ordinary second order elliptic problem considered in Chapter 3. For the general case of $1 < p < n$, several studies have appeared recently. We mention the work in bounded domain cases by Azorero and Alonso [AA], Egnell [Eg1, Eg2], Guedda and Veron [GV], Kichenassamy and Veron [KV], Knaap and Peletier [KP], Veron [Ve] and references therein. In unbounded domain cases, we recall the results of Bidaut-Veron [BV], Kura [Ku], Li and Yan [LY], Ni and Serrin [NiS1, NiS2]. Since we are interested in the problems in unbounded domains, we give more details about these papers. In [NiS1] and [NiS2], the existence and nonexistence of (IV) in the radial case, i.e., $\Omega = R^n$, $a \equiv 1$, $b \equiv 0$, $f(x, u) = f(u)$ were studied. In particular, Ni and Serrin showed that the equation

$$\operatorname{div}(|\nabla u|^{p-2} u) + u^{\alpha} = 0 \quad \text{in } R^n$$

admits no positive radial ground state solutions if $0 < \alpha < p^* - 1$, and conversely, the equation does admit one if $\alpha \geq p^* - 1$, see Section 6 of [NiS1]. Li and Yan considered two eigenvalue problems of the following prototype:

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) + \lambda |u|^{p-2} u - f(x, u) = 0$$

and

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) + \lambda(|u|^{p-2} u - f(x, u)) = 0$$

in R^n, and obtained a decaying solution u_λ in $W^{1,p}(R^n)$ for $\lambda = \lambda_0$ (some $\lambda_0 > 0$) under the assumption $\lim_{t \to 0} f(x, t)/t^{p-1} = 0$, by concentration compactness arguments. In [Ku], Kura established the existence of positive radial solutions bounded away from 0 for (IV) in the case $1 < p \leq 2,\ n \geq 3$ by the subsolution and supersolution approach. Guedda and Veron [GV], and Bidaut-Veron [BV] studied the local and global behaviour of solutions of (IV).

No existence theory seems to have been found to date for nonradially symmetric p-Laplacian problems of type (IV) in the cases (6.1.1), (6.1.2) and (6.1.3). Observe that the case $b(x) \equiv 0$ is the most interesting since the L^p-theory of (IV) in this case breaks down. Therefore, since $W_0^{1,p}(\Omega)$ is no longer a suitable space for (IV), new spaces must be found. As the p-Laplacian problem [illegible] most the same properties as the second order elliptic problems, we [illegible] apply the method introduced in Chapter 3 to problem (IV). The procedure follows the lines of Chapter 3. That is, we first set up some weighted spaces in which the solutions are to be sought and for which the norm $\|u\|_\ell^p = \int_\Omega a(x)|\nabla u|^p + b(x)|u|^p$ induced by the operator ℓ is an equivalent norm. We then employ Mountain Pass arguments to obtain the existence of solutions. To prove the decay of the solutions, we make use of the estimates of Serrin [Se] for quasilinear equations. Most of the proofs of this chapter are the same as those of Chapter 3. We only sketch the ideas and point out the differences here.

6.2. Superhomogeneous problem.

In this section we consider problem (IV) in the superhomogeneous case, that is, f is of prototype $g(x)t^\alpha$ with $p - 1 < \alpha < p^* - 1$.

The hypothesis on f are:

(1) $f \in C^0(\Omega \times R^+)$, $f(x,t) > 0$ in $\Omega_0 \times (0,\infty)$ for some open $\Omega_0 \subseteq \Omega$;

(2) $|f(x,t)| \leq g(x)|t|^\alpha$, $p-1 < \alpha < p^* - 1$, $(0 \not\equiv) g \in L^\infty \cap L^{p_0}(\Omega)$ where $p_0 = \frac{np}{np-(\alpha+1)(n-p)}$;

(3) There exists $\mu > p$ such that $\mu F(x,t) \leq tf(x,t) \quad (x,t) \in \Omega \times R^+$.

We choose the function space E as the completion of $C_0^\infty(\Omega)$ under the norm:

$$\|u\| = \left(\int_\Omega |\nabla u|^p + \omega_0 |u|^p\right)^{\frac{1}{p}}$$

where $\omega_0(x) = \max\left\{b(x),\ \sigma(x)^{\frac{p}{2}}\right\}$ and $\sigma(x)$ is defined in Chapter 2. From the definition, it is clear that $E \sim W_0^{1,p}(\Omega)$ if $b(x) \geq b_0 > 0$. Moreover, E has the following three important properties:

(a) E can be embedded into $W_{\text{loc}}^{1,p}(\Omega)$;

(b) Sobolev Inequality: $\|u\|_{p^*} \leq C\|\nabla u\|_p$ for all $u \in E$, $p^* = \frac{np}{n-p}$.

(c) The norm $\|u\|_\ell = \left(\int_\Omega a|\nabla u|^p + b|u|^p\right)^{\frac{1}{p}}$ induced by the operator ℓ is an equivalent norm in E.

Indeed, (a) and (b) are obvious, while (c) follows from a Hardy-type inequality

$$\int_\Omega \sigma^{\frac{p}{2}} |u|^p \leq C \int_\Omega |\nabla u|^p$$

for all $\varphi \in C_0^\infty(\Omega)$. The inequality can be obtained by applying the Divergence Theorem and Hölder inequality:

$$\begin{aligned}\int_\Omega \sigma^{\frac{p}{2}}|u|^p &= -\frac{1}{n}\int x\cdot\nabla\left(\sigma^{\frac{p}{2}}|u|^p\right)\\ &= -\frac{p}{n}\int_\Omega x\cdot\sigma^{\frac{p}{2}}|u|^{p-1}\nabla u + \frac{p}{n}\int_\Omega \frac{|x|^2}{1+|x|^2}\sigma^{\frac{p}{2}}|u|^p\\ &\leq \frac{p}{n}\int_\Omega\left(\sigma^{\frac{p-1}{2}}|u|^{p-1}\right)(|\nabla u|) + \frac{p}{n}\int_\Omega \sigma^{\frac{p}{2}}|u|^p.\end{aligned}$$

We now define two functionals $K(u)$ and $J(u)$ on E:

$$K(u) = \int_\Omega F(x,u)dx \tag{6.2.1}$$

$$\begin{aligned}J(u) &= \frac{1}{p}\|u\|_\ell^p - K(u)\\ &= \frac{1}{p}\int_\Omega a|\nabla u|^p + b|u|^p - \int_\Omega F(x,u).\end{aligned} \tag{6.2.2}$$

$K(u)$ is well-defined by the assumption (2) and Sobolev's Inequality. Observing the embedding $E \hookrightarrow W^{1,p}(\Omega_k)$ and the compact embedding $W^{1,p}(\Omega_k) \hookrightarrow L^q(\Omega_k)$ for $1 < q < p^*$ and bounded subdomain Ω_k, we can prove, in a similar way to the proof of Lemma 3.1 of Chapter 3, that K and J are weakly lower semicontinuous and differentiable in E with

$$K'(u)(\varphi) = \int_\Omega f(x,u)\varphi\ dx \tag{6.2.3}$$

and

$$J'(u)(\varphi) = \int_\Omega (a|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi + b|u|^{p-2}u\varphi - f(x,u)\varphi)dx, \tag{6.2.4}$$

and $K'(u)(\cdot)$ is a continuous and compact map from E to E^* - the dual of E.

We can now employ the Mountain Pass Theorem to obtain a solution of (IV).

THEOREM 6.1. *Under condition (1) - (3), the problem (IV) has a positive decaying solution in* $C^{1,\alpha}(\overline{\Omega})$.

PROOF: Since we seek positive solutions, it is convenient to define $f(x,t) = 0$ for $t \leq 0$. By condition (2),

$$J(u) \geq \frac{1}{p}\|u\|_\ell^p - C\|g\|_{p_0}\|u\|_\ell^{\alpha+1} \geq a$$

for $u \in B_r(0)$, some $r,\ a > 0$. From conditions (1) and (3), integrating shows $F(x,t) \geq a_1 t^\mu - a_2$ for $(x,t) \in \Omega_0 \times R^+$, some a_1, $a_2 > 0$. [illegible] $C_0^\infty(\Omega_0)$, pick $w(x) \geq 0,\ \not\equiv 0$ and let $s \in R^+$. For s large

$$\begin{aligned} J(sw) &\leq \frac{1}{p}s^p\|w\|_\ell^p - s^\mu \int_{\Omega_0} a_1 w^\mu + a_2|\Omega_0| \\ &< 0. \end{aligned}$$

Thus we obtain the existence of e with $J(e) < 0$. To see that the (PS) condition holds, suppose $\{u_i\} \subseteq E$ is such that

$$J(u_i) \leq C \quad \text{and} \quad J'(u_i)(\cdot) \to 0.$$

Note that the inequality:

$$\begin{aligned} C \geq J(u_i) &\geq \frac{1}{p}\|u_i\|_\ell^p - \frac{1}{\mu}\int_\Omega f(x,u_i)u_i \\ &\geq \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_i\|_\ell^p + \frac{1}{\mu}J'(u_i)(u_i), \end{aligned}$$

yields the boundedness of $\{u_i\}$. Consequently it follows from the compactness of K' that there exists a subsequence of $\{u_i\}$, say $\{u_i\}$ itself, such that $K'(u_i)$ is Cauchy in E^*. We claim that $\{u_i\}$ is a Cauchy sequence in E. Indeed, we have the inequality

$$|\nabla u_i-\nabla u_j|^p \le \begin{cases} (|\nabla u_i|^{p-2}\nabla u_i - |\nabla u_j|^{p-2}\nabla u_j)\cdot(\nabla u_i - \nabla u_j) & \text{if} \quad p \ge 2 \\ [(|\nabla u_i|^{p-2}\nabla u_i - |\nabla u_j|^{p-2}\nabla u_j)\cdot(\nabla u_i - \nabla u_j)]^{\frac{p}{2}}[|\nabla u_i|^p + |\nabla u_j|^p]^{\frac{2-p}{2}} & \\ \qquad\qquad \text{if} \quad 1 < p < 2 & \end{cases}$$

(see, e.g. [KV] and [Th2]). On the other hand, from (6.2.4) we obtain

$$\begin{aligned} &\int_\Omega \{a(|\nabla u_i|^{p-2}\nabla u_i - |\nabla u_j|^{p-2}\nabla u_j)\cdot(\nabla u_i - \nabla u_j) \\ &+ b(|u_i|^{p-2}u_i - |u_j|^{p-2}u_j)(u_i - u_j)\} \\ &\le |J'(u_i)(u_i - u_j)| + |J'(u_j)(u_i - u_j)| \\ &+ |\int_\Omega (f(x,u_i) - f(x,u_j))(u_i - u_j)| \\ &\le C\{\|J'(u_i)\|_{E^*} + \|J'(u_j)\|_{E^*} + \|K'(u_i) - K'(u_j)\|_{E^*}\}, \end{aligned}$$

where $C = C(n,p)$. It follows immediately that $\{u_i\}$ is Cauchy in E. Thus the (PS) condition holds. The Mountain Pass Theorem guarantees the existence of a nontrivial critical point of J', say u. Observe that u is a weak solution of $\ell u = f(x,u)$, i.e.

$$\int_\Omega a|\nabla u|^{p-2}\nabla u\nabla\varphi + b|u|^{p-2}u\varphi = \int_\Omega f(x,u)\varphi, \qquad (6.2.5)$$

for all $\varphi \in E$. Letting $\varphi = u^-(=\min\{u,0\})$ yields that $u \geq 0$ in Ω. Next we prove the decay of u. Set $u_k(x) = \min\{u(x), k\}$, $k = 1, 2, \ldots$. For any real $i \geq 1$, $(u_k)^i \in E$. Let us now substitute in (6.2.5) $\varphi = (u_k)^i$. The result is

$$\int_\Omega (u_k)^{i-1}|\nabla u_k|^p \leq \frac{\|g\|_\infty}{a_0 i}\int_\Omega u^{\alpha+i}.$$

By the fact that $(u_k)^{i-1}|\nabla u_k|^p = (\frac{p}{i+p-1})^p|\nabla(u_k)^{\frac{i+p-1}{p}}|^p$ and Sobolev's Inequality, we have

$$\left(\int_\Omega (u_k)^{\frac{n}{n-p}(i+p-1)}\right)^{\frac{n-p}{n}} \leq C\int_\Omega u^{\alpha+i}, \tag{6.2.6}$$

where $C = C(n, i, p, \alpha, a_0\|g\|_\infty)$. Setting

$$\sigma = p^* - 1 - \alpha,\ i = i_0 = 1 + \sigma,\ q_0 = \frac{n}{n-p}(i_0 + p - 1) = \frac{n}{n-p}(p + \sigma)$$

and letting $k \to \infty$ in (6.2.6), we conclude $u \in L^{q_0}(\Omega)$. As we did in the proof of Lemma 3.2, Chapter 3, we can prove by iterating the process that $u \in L^q(\Omega)$ for $p^* \leq q < \infty$. The compactness of $\partial\Omega$ guarantees that for all x such that $|x| \geq r_0$, some $r_0 > 0$, $B_2(x) \subseteq \Omega$. Then by Theorem 1 of Serrin [Se], for some $q > \frac{n}{p}$.

$$\|u\|_{L^\infty(B_1(x))} \leq C\{\|u\|_{L^{p^*}(B_2(x))} + \|f(x,u)\|_{L^q(B_2(x))}\}$$

where $C = C(n, p, q)$. Thus the decay of u follows. The $C^{1,\alpha}$ boundary and interior regularities follow basically from [To1] and [To2] while the positivity follows from [Va].

6.3. Other Problems.

With some modifications we employ the method of the previous section to consider two other problems: the subhomogeneous problem and the mixed sub-superhomogeneous problem.

We introduce the following conditions:

(4) $0 \leq f(x,t) \leq h(x)|t|^{\beta}$, $0 \leq \beta < p-1$, $h \in L^{\infty}(\Omega) \cap L^{q_0}(\Omega)$, $q_0 = \frac{np}{np-(\beta+1)(n-p)}$;

(5) $f(x,t) \geq h_0(x)t^{\beta_0}$ as $t \to 0^+$, $0 \leq \beta_0 \leq \beta$, $h_0(x) \geq 0$, $\not\equiv 0$.

THEOREM 6.2. *Under conditions (1), (4) and (5), problem (IV) has a positive decaying solution in* $C^{1,\alpha}(\overline{\Omega})$.

PROOF: We simply sketch the idea since the proof is similar to that of Theorem 3.4. We assume $f(x,t) = f(x,0)$ for $t \leq 0$. By condition (4), the functional J is still weakly lower semicontinuous and bounded below. Thus J has a criti[illegible] which is a solution of (IV). Condition (5) guarantees that u is nontrivial. The proof of the decay used in Theorem 6.1 works here also.

For the mixed sub-superhomogeneous problem, we have

THEOREM 6.3. *Let* $f(x,t) = f_1(x,t) + f_2(x,t)$, *suppose that* f_1 *satisfies (1) - (3) and* f_2 *satisfies (1) and (4). Then the problem (IV') has a positive decaying* *solution in* $C^{1,\alpha}(\overline{\Omega})$ *provided*

$$
\begin{aligned}
2B^p\|g\|_{p_0}^{\frac{p-\beta-1}{\alpha-\beta}} \cdot \|h\|_{q_0}^{\frac{\alpha+1-p}{\alpha-\beta}} \Big[& \frac{1}{\alpha+1}\left(\frac{(\alpha+1)(p-\beta+1)}{(\beta+1)(\alpha+1-p)}\right)^{\frac{\alpha+1-p}{\alpha-\beta}} \\
& + \frac{1}{\beta+1}\left(\frac{(\beta+1)(\alpha+1-p)}{(\alpha+1)(p-\beta-1)}\right)^{\frac{p-\beta-1}{\alpha-\beta}}\Big] \\
< 1, &
\end{aligned}
$$

where B *is the embedding constant for* $C_0^\infty(\Omega) \hookrightarrow L^{p^*}(\Omega)$.

PROOF: The proof is very similar to that of Theorem 3.5, with the only differences being the replacements of 2 by p, A_1 by B, etc. We omit it.

CHAPTER 7

CONCLUSION

In this thesis we have studied elliptic equations on unbounded domain in four directions:

(a) second order elliptic problems;

(b) higher order elliptic problems;

(c) second order elliptic eigenvalue problems;

(d) p-Laplacian problems,

and established existence results for these problems. We were primarily interested in positive solutions which decay at ∞ of nonradial problems. Because of the variational structure of these problems, we made use of critical point theory to show the existence of solutions.

For the superlinear problem and mixed sub-superlinear problem, it was natural to apply the Mountain Pass Theorem, as it is a powerful tool in the case of bounded domains. The difficulties encountered in passing to the unbounded domains were overcome by:

(i) the suitable choice of the function space E in which the solutions were sought;

(ii) the establishment of the compactness of the potential map $\int_\Omega f(\cdot, u)(\cdot)dx$ from E to E^* (the dual of E).

With the aid of two Hardy-type inequalities (Lemma 2.1) we set up the space E depending on the operator ℓ. Most of the properties of E are the same as those of $W_0^{m,2}(\Omega)$ due to the local embedding from E to $W_0^{m,2}(\Omega)$. Under certain integration conditions on $f(x,t)$, we obtained the compactness of $\int_\Omega f(\cdot,u)(\cdot)dx$, and the existence of the solutions for the sublinear, superlinear and mixed sub-superlinear problems followed. To show the decay of the solutions, we made use of standard L^p estimates (Theorem 8.25 [GT] and Theorem 1 [Se]) for the second order elliptic problem and p-Laplacian problem, and of Agmon's L^p estimates (Theorem 6.1 and Theorem 6.2 [Ag]) for higher order elliptic problems. Note that the latter approach is also valid for the second order case.

As our methods do not use radial arguments, our results can apply in general cases with highly nonradial properties. Our condition in the superlinear case is optimal to some extent as shown by a nonexistence result of Kusano and Naito (Example 2, Section 3.4). In the direction of higher order elliptic sublinear and superlinear problems, our conditions are better than the earlier ones. We gave a complete answer to an open question in the superlinear case of Kusano, Naito and Swanson ([KNS3]). For the elliptic eigenvalue problem, we extended the work in the bounded domain case of Brown and Budin [BB] and Hess [He] to the unbounded domain case. The p-Laplacian problem is also considered under nonradial conditions. To our knowledge, no previous results had been obtained in this direction.

We point out, however, that our existence result for the second order elliptic problem in the sublinear case (Theorem 3.4, Chapter 3) is not optimal.

Specifically, if $f(x,u) = g(x)u^\beta$, $0 \le \beta < 1$ with $g(x) \sim |x|^{-a}$, our condition (4) in Chapter 3 requires $a > \frac{1}{2}(n + 2 - \beta(n-2))$, while the earlier result in the radial case by Fukagai [Fu1] requires $a > 2$. Our condition is stronger especially for large n and small β.

In this thesis, we did not discuss the total mass property $\int_\Omega f(x,u)dx < \infty$ of the solutions for second order elliptic problems. In [LN2], Li and Ni proved some existence and nonexistence results of finite total mass solutions for the Matukuma and Eddington equations. It would be interesting to obtain this property for the general case. The difficulty is that there are no estimates of the decay speed of the solutions for nonradially symmetric elliptic problem. Due to the restriction of our methods, we only studied the problems with variational structure. The study of decaying solutions of problems with nonvariational structure is of interest, and not much is known. For instance, it would be interesting to extend known results to the following problem:

$$\begin{cases} -\sum D_i(a_{ij}(x)D_ju) + \sum b_iD_iu + cu = f(x,u), & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \quad \lim\limits_{|x|\to\infty} u = 0, \end{cases}$$

or the more general one,

$$\begin{cases} -\sum D_i(a_{ij}(x)D_ju) + cu = f(x,u,\nabla u) & \text{in } \Omega \\ u|_{\partial\Omega} = 0, \quad \lim\limits_{|x|\to\infty} u = 0. \end{cases}$$

Finally, we mention two other open problems: first, to find criteria for the existence of positive solutions of higher order problems in R^n, n small; second, to find criteria for the existence of positive solutions to such problems in general $\Omega \subset R^n$. We plan to investigate these problems in the future.

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